On tensor induction for representations of finite groups

ON TENSOR INDUCTION FOR REPRESENTATIONS OF FINITE GROUPS

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Abstract. It is well known that if D is an irreducible complex representation of a finite group G, then every direct summand of the restriction of D to a subgroup H must have degree at least as large as the degree of D divided by the index $|G:H|$; moreover, D is induced from H if and only if the restriction does have a direct summand whose dimension is equal to this quotient. This paper explores the possibility of an analogous result for tensor induction, under the additional assumption that D is faithful, quasi-primitive and not a tensor product (of projective representations of degree greater than 1), and that the Fitting subgroup $F(G)$ is not in the center $Z(G)$. The main question is this: if the restriction has a (projective) tensor factor whose degree is the $|G:H|$th root of the degree of D, does it follow that D is tensor induced from H? Among other results, examples are given to show that the answer can be negative when the index is 2. An affirmative answer is proved for normal subgroups of odd index, and also for arbitrary subgroups of odd prime index. As might be expected, the key lies in the study of $F(G)/Z(G)$ as a symplectic module over a finite prime field; in particular, in exploring the connection between (ordinary) induction and form-induction of such modules.

Key words and phrases. Representations of finite groups, tensor factorization, tensor induction, symplectic modules.

Introduction

The general aim of this work is to analyse, from a particular point of view, the 'multiplicative structure' of quasi-primitive complex representations of finite groups. All the abstract groups considered throughout the paper are meant to be finite. All the representations will be finite dimensional and, whenever no explicit indication is given, over the complex field.

I. An irreducible representation of a group is called quasi-primitive if its restriction to every normal subgroup is homogeneous, that is, it has pairwise equivalent irreducible constituents. As it is well known, given an irreducible representation D of a group G, Clifford's Theorem ([**3**, 11.1]) enables us to 'recognize' a subgroup H of G, and a quasi-primitive representation T of H, such that D *is induced by* T *from* H; in other words, D can be constructed in a well understood way (which exploits the additive structure of representations) by means of T, and in this sense

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quasi-primitive representations may be viewed as 'building blocks' for irreducible representations.

As there seems to be no general method to exploit further the additive structure of quasi-primitive representations (recall also that, by a theorem of Berger [**10**, 11.33], in the context of solvable groups quasi-primitivity is the same as primitivity), it appears convenient to investigate these objects from a multiplicative point of view. More explicitly, given a group G and a quasi-primitive representation D of G, it is relevant to understand if and how D can be decomposed as a tensor product of projective representations of G (tensor factorization for quasi-primitive representations is studied, for instance, in [**4**] and [**13**]).

The subsequent step, and the starting point for the present discussion, is to focus on quasi-primitive representations which do not admit any tensor factorization (see Definition 3.1). Let G be a group which has a faithful, quasi-primitive, and tensor-indecomposable representation D; the existence of such a representation implies a strong restriction on the structure of G. Namely, denoting by F the Fitting subgroup (which is assumed noncentral) of G, and by Z the centre of G, the section F/Z turns out to be itself a simple G-module over a prime field (see Lemma 3.5(a)). Moreover, the module F/Z carries a nonsingular symplectic form which is G-invariant. This is the natural context in which the method of *tensor induction* plays a significant role (tensor induction is a process defined in analogy with ordinary induction, and it can be thought as a transposition of it to a multiplicative context; see Section 1). In particular, we can ask whether the representation D is tensor-induced by a projective representation of a proper subgroup H of G. The deep link between tensor induction for D and the *additive* structure of the symplectic module F/Z is observed and discussed by T. R. Berger in [**1**] and in [**2**]. Moreover, a theorem by L. G. Kovács ([**11**, Section 6]) can be paraphrased by saying that D is tensor-induced by a projective representation of H if F/Z is *form-induced* from H (form induction is a kind of ordinary induction in which also the structure given by the symplectic form is taken in account; see Definition 3.3). Theorem 3.7 in this paper completes the picture, proving that, if D is a faithful, quasi-primitive, tensor-indecomposable representation of G, and if H is a subgroup of G, then *a bijection can be constructed between the set of (equivalence classes of) projective representations of H which tensor-induce D, and the set of H-submodules of F/Z which form-induce F/Z.*

II. We mentioned that a significant analogy can be established between ordinary induction and tensor induction, and in general between additive and multiplicative methods in the representation theory of finite groups. Nevertheless, as we shall see, such an analogy is far from being complete. We present next the main problem discussed in paper; the analysis of it will emphasize several aspects of this incompleteness.

The concept of ordinary induction is closely related to the concept of *restriction*; indeed, from the point of view of modules and categories, ordinary induction may be characterized as adjoint to restriction (on both sides). As a consequence of this fact, we have the following well known 'reciprocity'. Let D be an irreducible representation of G, and T a representation of the subgroup H; then D is induced by T from H if and only if T appears as a direct summand in the restriction $D{\downarrow_H}$ and $\deg D = |G:H| \deg T$. Since tensor induction may be interpreted as the multiplicative counterpart of ordinary induction, one could try to formulate a parallel statement, as follows.

Let D be a faithful, quasi-primitive and tensor-indecomposable representation of G, and P a projective representation of the subgroup H. Then D is tensor-induced by P from H if and only if P appears as a tensor factor in $D{\downarrow_H}$ and $\deg D = (\deg P)^{|G:H|}$.

In a weaker form, one may ask whether (in the same setting) D *is tensor-induced from H if and only if $D{\downarrow_H}$ has a tensor factor of degree* $(\deg D)^{1/|G:H|}$ (D not being tensor-induced necessarily by *that* tensor factor).

This question has been proposed by L. G. Kovács, and the core of the present work consists in studying it, mainly in the context of solvable groups (the statement above appears in the paper as Conjecture 4.1, which has a strong and a weak version); the original motivation for this kind of analysis is to seek a way of characterizing tensor induction in an 'internal' fashion (similarly to what happens for ordinary induction), and to clarify the relationship between tensor induction and restriction. Moreover, as Theorem 10.3 shows, even the weak version of Conjecture 4.1 (in the cases in which it is confirmed) provides a good test for tensor induction, since the results of [**13**] control all the possible factorizations of $D{\downarrow_H}$. We also mention that some effective computational algorithms for the internal recognition of tensor products (and of tensor induction) have been developed and implemented by C. R. Leedham-Green and E. A. O'Brien in [**12**].

Our approach to Conjecture 4.1 consists of a chain of subsequent reductions, which may be outlined as follows. First of all, the characterization of tensor induction achieved in Section 3 enables us to transpose the problem into a more accessible additive setting. Indeed, Conjecture 4.1 appears to be deeply linked to a statement (Conjecture 4.3) which establishes a connection between ordinary induction and form induction for symplectic modules *over finite fields*. Conjecture 4.3 presents a strong and a weak version as well.

At this level, it is already possible to show that Conjectures 4.1 and 4.3 both fail in their strong version, and also in the weak one if the index of the subgroup H in G is not assumed to be odd (this is achieved in Examples 5.1 and 5.2). What is then left is to concentrate on the weak versions of the conjectures, with the additional assumption that $|G:H|$ is odd.

The successive reductions are obtained through an analysis of the relationship between modules and bilinear forms, which is developed in Sections 6 and 7 (Remark 7.5 and Lemma 7.7 are particularly important). Given that, we are in a position to obtain positive results. Namely, the weak versions of Conjectures 4.1 and 4.3 are proved to be true for *normal* subgroups of odd index (see Theorems 8.2 and 8.3), and this, together with Example 5.2, provides a full understanding of what happens in this context with respect to normal subgroups. As regards subgroups which are not necessarily normal, Theorems 10.1 and 10.2 show that the two (weak) conjectures are true for subgroups whose index is an (odd) prime. These are proved after two crucial results (Theorems 9.7 and 9.10), concerning the structure of modules induced from maximal subgroups, are established. At this stage it is also worth remarking that Example 5.1, which disproves the strong versions of 4.1 and 4.3, involves a normal subgroup of odd prime index.

What remains to be understood is whether Conjectures 4.1 and 4.3 are valid for not normal subgroups whose index is odd, but not necessarily a prime. This is left as an open problem. However, extending Theorem 9.7 (if possible) to subgroups having index *a power of* an odd prime could be a decisive step in order to achieve the final answer.

To conclude, the last section of the paper contains an example (11.1) whose aim is to clarify some issues arising from Sections 7 and 9. Moreover, Example 11.1 appears as a 'summary' of awkward behaviours of form induction (with respect to ordinary induction), and it illustrates unexpected situations in the structure of induced modules.

1. Projective representations and tensor induction

Let G be a group, $\mathbb{F}$ a field, and d a positive integer; a map $P : G \to GL(d,\mathbb{F})$ is called a projective representation of G (of degree d, over the field $\mathbb{F}$) if the map $\bar{P}$, defined as the composite of P with the natural homomorphism of $GL(d,\mathbb{F})$ onto $PGL(d,\mathbb{F})$, is a homomorphism. Such a P is called irreducible if the preimage of $\bar{P}(G)$ in $GL(d,\mathbb{F})$ is an irreducible linear group. Since any representation is clearly also a projective representation, sometimes, for the sake of emphasis, a representation in the classical sense will be referred to as a *genuine* representation.

If P_1 and P_2 are projective representations of G having the same degree d, then they are called equivalent if $\bar{P}_2$ is the composite of $\bar{P}_1$ with an inner automorphism of $PGL(d,\mathbb{F})$. In this case we write $\bar{P}_1 \simeq \bar{P}_2$; also, we denote by $[P]$ the equivalence class of a projective representation P modulo the equivalence relation which arises in this way.

Projective representations are important in the present context because it is possible to define the concept of inner tensor product (and, consequently, of tensor induction) for them, and such a product may yield a genuine representation. If P_1 and P_2 are projective representations of G over $\mathbb{F}$, having degrees c and d respectively, then the map $P_1 \otimes P_2 : G \to GL(cd,\mathbb{F})$, defined by $(P_1 \otimes P_2)(g) := P_1(g) \otimes P_2(g)$ for all g in G, is a projective representation of G (the symbol '$\otimes$' denotes here the usual Kronecker product of matrices); this projective representation is called the inner tensor product of P_1 and P_2.

In view of the fact that two different concepts of equivalence are defined for genuine representations (depending on whether they are regarded as genuine or as projective representations), it is sometimes convenient to emphasize the distinction, speaking of the *genuine-equivalence type* or the *projective-equivalence type* of a representation. It is clear that, if D_1 and D_2 are genuine representations of G, then $\bar{D}_1 \simeq \bar{D}_2$ holds if and only if there exists a 1-dimensional representation λ of G such that D_2 and $\lambda \otimes D_1$ are genuine-equivalent.

We recall next the concept of tensor induction, which is really central in our discussion. Let H be a subgroup of G having index n, and P a projective representation of H; tensor induction, applied to P, yields a projective representation of G whose degree is $(\deg P)^n$. Some preparation is needed in order to give the precise definition.

First, let π denote the usual permutation representation of G on the set Ω of right H-cosets (the action of G on Ω is given by right multiplication), and let $\{g_1, \dots, g_n\}$ be a right transversal for H in G. For any x in G and i in $\{1, \dots, n\}$, let $h(i,x)$ be the (uniquely determined) element of H such that $g_i x = h(i,x) g_{i(x\pi)}$ holds; if now $\varphi : G \to H \wr S_n$ is the map defined by $x\varphi := (h(1,x), \dots, h(n,x))(x\pi)$ for all x in G, then φ turns out to be a monomorphism of groups (see [**3**, 13.3]). Next, following [**11**], we denote by $P \wr S_n : H \wr S_n \to GL(\deg P, \mathbb{F}) \wr S_n$ the map which associates an element $(h_1, \dots, h_n)\sigma$ to $(P(h_1), \dots, P(h_n))\sigma$, and by $k : GL(\deg P, \mathbb{F}) \wr S_n \to GL((\deg P)^n, \mathbb{F})$ the homomorphism whose restriction to the base group is given by the n-fold Kronecker product, whereas the restriction of k to the top group is an isomorphism to one distinguished subgroup of $GL((\deg P)^n, \mathbb{F})$: the elements of such a subgroup are permutation matrices, whose action by conjugation on the image of the base group corresponds to permuting the Kronecker factors.

We are now in a position to give the definition of tensor-induced representation.

Definition 1.1. Let G be a group, H a subgroup of G of index n, $\mathbb{F}$ a field, and P a projective representation of H over $\mathbb{F}$. We define $P{\uparrow}^{\otimes G}\colon G \to GL((\deg P)^n, \mathbb{F})$ as the composite map $\varphi(P \wr S_n)k$. Such a map is a projective representation of G, whose equivalence type is not affected by any of the (several) choices involved in the defining process (for instance, a right transversal for H in G has to be fixed). Also, if P' is a projective representation of H equivalent to P, then $(P'){\uparrow}^{\otimes G}$ is equivalent to $P{\uparrow}^{\otimes G}$ as well. *Any projective representation of G which is equivalent to $P{\uparrow}^{\otimes G}$ is said to be tensor-induced by P from H.*

It is clear that, if T is a genuine representation of H, then the tensor-induced representation $T{\uparrow}^{\otimes G}$ is also genuine. In order to emphasize that tensor induction may be viewed as the multiplicative transposition of ordinary induction of genuine representations, we remark that the latter can be defined in close analogy with 1.1. Indeed, the induced representation $T{\uparrow}^G$ can be defined as the composite map $\varphi(T \wr S_n)m$, where $m : GL(\deg T, \mathbb{F}) \wr S_n \to GL(n \deg T, \mathbb{F})$ is the usual 'block monomial' embedding. Nevertheless, as observed in [**11**], while there are alternative ways to define induction which are completely 'choice-free', nothing similar seems to be known so far for tensor induction.

We conclude the section introducing some more notations.

Definition 1.2. Let G be a group, and H a subgroup of G. If P_1 and P_2 are projective representations of G, then we denote by $\bar{P}_1 \otimes \bar{P}_2$ the homomorphism

$\overline{P_1 \otimes P_2}$. If P is a projective representation of H, then we denote by $\bar{P}{\uparrow}^{\otimes G}$ the homomorphism $\overline{P{\uparrow}^{\otimes G}}$.

2. Two results on tensor factorization

For later use, we recall the statements of two results concerning tensor factorization of complex representations. Proofs may be found in [**13**, 2.5 and 2.8].

Lemma 2.1. *Let F be a group whose centre Z is cyclic, and such that F/Z is abelian of squarefree exponent; then the following properties hold:*

(a) *if K is a subgroup of F such that $Z(K) = Z$, then F is the (central) product of K and $C_F(K)$;*

(b) *if P is an irreducible projective representation of F with $Z \leq \ker \bar{P}$, then we have $(\deg P)^2 = |F : \ker \bar{P}|$;*

(c) *if D is a faithful irreducible representation of F, and $\bar{D} \simeq \bar{P}_1 \otimes \bar{P}_2$ where P_1 and P_2 are projective representations of F, then we have $F = \ker \bar{P}_1 \cdot \ker \bar{P}_2$;*

(d) *with the same assumptions as in* (c)*, if K is the kernel of $\bar{P}_1$, then $Z(K)$ coincides with Z; moreover, denoting by L the kernel of $\bar{P}_2$, we have $L = C_F(K)$.*

Theorem 2.2. *Let G be a group with centre Z and Fitting subgroup F, and H a subgroup of G containing F; also, let D be a faithful quasi-primitive representation of G such that $D{\downarrow}_F$ is irreducible. There is a bijection between the set of all the pairs $([P_1], [P_2])$, where P_1 and P_2 are projective representations of H such that $\bar{D}{\downarrow}_H \simeq \bar{P}_1 \otimes \bar{P}_2$, and the set of normal subgroups K of H such that $K \leq F$ and $Z(K) = Z(H)$ hold. In particular, such a bijection can be constructed by mapping $([P_1], [P_2])$ to $K := \ker(\bar{P}_1{\downarrow}_F)$.*

Observe that, if in 2.2 the subgroup H is the whole G, then the second set involved in the mentioned bijection is precisely the interval $[Z, F]$ in the lattice of normal subgroups of G.

3. Form induction for symplectic modules and tensor induction

In this section we start our analysis of tensor induction for quasi-primitive representations. As mentioned in the Introduction, we focus on representations for which the process of tensor factorization yields no reduction; more precisely, we shall be dealing with *tensor-indecomposable* representations, in the following sense.

Definition 3.1. Let G be a group, and D a representation of G. Then D is called tensor-indecomposable if there do not exist two projective representations P_1 and P_2 of G, whose degrees are greater than 1, such that $\bar{D} \simeq \bar{P}_1 \otimes \bar{P}_2$.

Let G be a group, and D a faithful, quasi-primitive, tensor-indecomposable representation of G; assume also that a subgroup H of G is fixed. Our first aim, to which this section is devoted, is to achieve a 'parametrization' of all the ways in which D can be tensor-induced by a projective representation of H. As such a characterization is obtained in terms of the additive structure of a particular symplectic module (which 'comes' from the group structure of G), we recall next some relevant definitions.

Definition 3.2. Let G be a group, $\mathbb{F}$ a field, V an $\mathbb{F}G$-module, and f a bilinear $\mathbb{F}$-form defined on (the underlying vector space of) V; if $f(u^g, v^g) = f(u, v)$ holds for all u, v in V and g in G, then f is called G*-invariant*.

Definition 3.3. Let G be a group, H a subgroup of G, $\mathbb{F}$ a field, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow_H}$. Assume that a G-invariant nonsingular symplectic $\mathbb{F}$-form f is defined on V (in this case, V is also called a *symplectic module* with respect to f); assume also the following conditions:

(a) the restriction of f to $W \times W$, which is an H-invariant symplectic $\mathbb{F}$-form on W, is nonsingular;
(b) the translate W^g lies in $W^{\perp}$ for all g in G such that $W^g \neq W$;
(c) V is induced by W from H.

Then we say that V is *form-induced* by W (with respect to f) from H.

In what follows we shall take advantage from the 'Tensor Induction Theorem' of L. G. Kovács ([**11**, Section 6]), which is stated in a partial and paraphrased form as Theorem 3.4; the subsequent lemma will provide, besides other information, a converse for it.

Theorem 3.4. *Let G be a (not necessarily finite) group with a faithful, quasi-primitive and tensor-indecomposable representation D. Let L be a noncentral subgroup which centralizes all its conjugates in G except perhaps itself, and set $N_G(L) = H$. Then D is tensor-induced by a projective representation P of H such that* $\ker \bar{P} = C_G(L)$.

Lemma 3.5. *Let G be a group with centre Z and noncentral Fitting subgroup F, and let D be a faithful, quasi-primitive, tensor-indecomposable representation of G. Assume that H is a subgroup of G of index n, $\{g_1 = 1, \ldots, g_n\}$ is a right transversal for H in G, and P is a projective representation of H such that $\bar{D} \simeq \bar{P}{\uparrow^{\otimes G}}$. Then the following properties hold.*

(a) *F/Z is a chief factor of G, therefore an elementary abelian p-group for a suitable prime p. As a $GF(p)[G]$-module (G acting by conjugation), F/Z carries a G-invariant nonsingular symplectic form which comes from taking commutators in F.*
(b) *H contains F, thus it is possible to define the subgroups* $K := \ker(\bar{P}{\downarrow_F})$ *and* $L := C_F(K)$.
(c) *If $Q : F \to GL((\deg P)^{n-1}, \mathbb{C})$ is the projective representation of F defined by $Q(y) := \bigotimes_{j=2}^{n} P(y^{g_j^{-1}})$ for all y in F, then we have*
$$\bar{D}{\downarrow_F} \simeq \bar{P}{\downarrow_F} \otimes \bar{Q}, \quad \ker \bar{Q} = L \quad \textit{and} \quad F = KL;$$
(d) *The $GF(p)[G]$-module F/Z is form-induced by L/Z from H, with respect to the form of* (a).
(e) *$P{\downarrow_L}$ is irreducible, and there exists an irreducible genuine representation S of L such that $\bar{P}{\downarrow_L} = \bar{S}$. Moreover, the projective-equivalence type of P is uniquely determined by L and the genuine-equivalence type of D.*

Proof of (a). Observe that, D being quasi-primitive and F being noncentral, the restriction of D to F is irreducible (otherwise, by a theorem of Clifford ([**3**, 11.20]), D would have a tensor decomposition). Applying Schur's Lemma, we see that $C_G(F)$ lies in Z, hence we get $Z(F) = Z$. Moreover, since G has a faithful quasi-primitive representation, F/Z is abelian of squarefree exponent (see for instance [**4**, 1.4]), so that F is a group as in the hypothesis of 2.1. Now, the assumption of tensor indecomposability for D, together with 2.2, yields that F/Z is a chief factor of G, hence a simple $GF(p)[G]$-module for a suitable prime p. Also, as F/Z is an elementary abelian p-group, we have that F' lies in Z and it has order p; therefore we can choose a generator (x say) for F', and define a map $\alpha : F \times F \to GF(p)$ by means of the following relations:

$$[y_1, y_2] = x^{\alpha(y_1,y_2)} \quad \text{for all } y_1, y_2 \text{ in } F.$$

If we now define a map $f : F/Z \times F/Z \to GF(p)$ by setting $f(Zy_1, Zy_2) := \alpha(y_1, y_2)$, then it is straightforward to check that f is a G-invariant nonsingular symplectic form on F/Z.

Proof of (b). As a general fact, we observe that the kernel of a homomorphism of the kind $\bar{P}{\uparrow}^{\otimes G}$ (P being a projective representation of H) is the normal core of $\ker \bar{P}$ in G (if z is in $\ker(\bar{P}{\uparrow}^{\otimes G})$, which means that $P{\uparrow}^{\otimes G}(z)$ is a scalar matrix, then z permutes trivially the right H-cosets in its action by right multiplication; hence z lies in $\mathrm{core}_G(H)$, and $P{\uparrow}^{\otimes G}(z)$ is given by $\bigotimes_{j=1}^{n} P(z^{g_j^{-1}})$. Now $P(z^{g_j^{-1}})$ is forced to be a scalar matrix for all j in $\{1, \dots, n\}$). In our context, this implies in particular that H contains Z. Consider now a minimal normal subgroup N/Z of G/Z; since F/Z is also minimal normal in G/Z, we have $[N, F] \leq Z$, so that $[N, F, N] = [F, N, N] = 1$. Applying the 'Three Subgroups Lemma' (see [**6**, 2.2.3]), we get $[N, N, F] = 1$, whence $[N, N] \leq Z$, and this means that N/Z is abelian. By the discussion above, it is clear that F/Z is the unique minimal normal subgroup of G/Z. If claim (b) were false, then H/Z would be core-free in G/Z, so that G/Z would be embedded in S_n. Now, $(\deg P)^n = \deg D$ is a divisor of $|G/Z|$, and this yields a contradiction, since $n!$ is not divisible by any n-th power.

Proof of (c). This is clear, since F is contained in the normal core of H in G, and therefore it acts trivially by right multiplication on the right H-cosets. The remaining claims follow immediately from parts (c) and (d) of Lemma 2.1.

Proof of (d). We know that, for any element l of L, the matrix $Q(l)$ is given by $P(l^{g_2^{-1}}) \otimes \cdots \otimes P(l^{g_n^{-1}})$, and it is a scalar matrix. This forces $L^{g_j^{-1}}$ to be contained in K for all j in $\{2, \dots, n\}$, hence we have $[L^{g_j^{-1}}, L] = 1$ and therefore $[L, L^{g_j}] = 1$ for all j in $\{2, \dots, n\}$. This means exactly that $(L/Z)^g$ is contained in $(L/Z)^{\perp}$, with respect to the symplectic form defined in (a), for all g with $(L/Z)^g \neq L/Z$. Applying now Lemma 2.1(b), we get $|L/Z| = |F/K| = (\deg P)^2$, and also $|F/Z| = (\deg D)^2 = (\deg P)^{2n}$; therefore L/Z is a submodule of $(F/Z){\downarrow}_H$ such that $n \dim(L/Z) = \dim(F/Z)$. This is sufficient to conclude that F/Z is induced by L/Z from H and, as the relevant symplectic form is clearly nonsingular on L/Z, the claim follows.

Proof of (e). Since the restriction of P to F is irreducible and $F = KL$, where K is the kernel of $\bar{P}{\downarrow_F}$, it is clear that $P{\downarrow_L}$ is irreducible. Recalling Definition 1.1, there exists an element A of $GL(\deg D, \mathbb{C})$ and a map λ of G to $\mathbb{C}^\times$ such that $\lambda(h)A^{-1}D(h)A = (P(h) \otimes X(h))Y(h)$ for all h in H, where $X(h)$ is in $GL((\deg P)^{n-1}, \mathbb{C})$, and $Y(h)$ is a permutation matrix (on n objects) in $GL(\deg D, \mathbb{C})$, which fixes (acting by conjugation) the first Kronecker factor. If l is an element of L, then we get $\lambda(l)A^{-1}D(l)A = P(l) \otimes \mu(l)I_{(\deg P)^{n-1}}$, where μ is a map of L to $\mathbb{C}^\times$. Defining now $S : L \to GL(\deg P, \mathbb{C})$ by $S(l) := \lambda(l)^{-1}\mu(l)P(l)$ for all l in L, we see that S is an irreducible genuine representation of L such that $\bar{P}{\downarrow_L} = \bar{S}$. From $D(l^h) = D(l)^{D(h)}$ we obtain now

$$S(l^h) \otimes I_{(\deg P)^{n-1}} = S(l)^{P(h)} \otimes I_{(\deg P)^{n-1}},$$

whence $S(l^h) = S(l)^{P(h)}$ for all h in H and l in L. Since the genuine-equivalence type of S is uniquely determined by L and by the genuine-equivalence type of D, it is not hard to see (using Schur's Lemma) that also the last part of claim (e) holds. ∎

After the next definition, we shall be in a position to prove the main result of this section, which shows that the concepts of form induction and tensor induction are related to each other (in the present context) in a very strong sense.

Definition 3.6. Let G be a group with centre Z and Fitting subgroup F; let H be a subgroup of G, and D a faithful, quasi-primitive, tensor-indecomposable representation of G. We define the sets

$$\mathcal{T}{\uparrow}_H^{\otimes G} := \{[P] : P \text{ is a projective representation of } H \text{ such that } \bar{D} \simeq \bar{P}{\uparrow}^{\otimes G}\}$$

and $\mathcal{F}{\uparrow}_H^G := \{L : Z \leq L \leq F \text{ and } F/Z \text{ is form-induced by } L/Z \text{ from } H\}$, where form induction is meant with respect to the symplectic form defined in Lemma 3.5(a).

Theorem 3.7. *Let G be a group with centre Z and noncentral Fitting subgroup F, and let H be a subgroup of G; also, let D be a faithful, quasi-primitive, tensor-indecomposable representation of G. There exists a bijection between the set $\mathcal{T}{\uparrow}_H^{\otimes G}$ and the set $\mathcal{F}{\uparrow}_H^G$. In particular, such a bijection can be constructed by mapping an element $[P]$ of $\mathcal{T}{\uparrow}_H^{\otimes G}$ to $L := C_F(\ker(\bar{P}{\downarrow_F}))$.*

Proof. Let L be an element of $\mathcal{F}{\uparrow}_H^G$; then L is a noncentral subgroup of G which centralizes all its conjugates in G except perhaps itself, and $N_G(L) = H$. We can therefore apply the Tensor Induction Theorem (3.4), and conclude that there exists a projective representation P of H such that $\bar{D} \simeq \bar{P}{\uparrow}^{\otimes G}$ and $\ker(\bar{P}{\downarrow_F}) = C_F(L)$. Since L is contained in $C_F(\ker(\bar{P}{\downarrow_F}))$, and both $C_F(\ker(\bar{P}{\downarrow_F}))/Z$ and L/Z induce F/Z from H, it is clear that $L = C_F(\ker(\bar{P}{\downarrow_F}))$. Now, by Lemma 3.5(e), the projective-equivalence type of P depends only on L and on the genuine-equivalence type of D, hence we can consistently define a map α from $\mathcal{F}{\uparrow}_H^G$ to $\mathcal{T}{\uparrow}_H^{\otimes G}$ setting $\alpha(L) := [P]$. Next, let $\beta : \mathcal{T}{\uparrow}_H^{\otimes G} \to \mathcal{F}{\uparrow}_H^G$ be the map defined by $\beta([P]) := C_F(\ker(\bar{P}{\downarrow_F}))$; from (b) and (d) of Lemma 3.5 it follows that this is a good definition and, since it is clear that the β so defined is a two-sided inverse to α, the result is proved. ∎

4. Two conjectures

As recalled in the Introduction, ordinary induction and restriction for genuine representations are deeply linked to each other, and one feature of such a good relationship is the fact that restriction provides a kind of 'internal characterization' for induction: if D is an irreducible representation of a group G, and T is a representation of a subgroup H of G, then $D \simeq T{\uparrow}^G$ holds if and only if T is an irreducible constituent of $D{\downarrow}_H$ with $|G:H|\deg T = \deg D$.

Our general aim is to seek an internal characterization of this kind for tensor induction. In view of the analogy between additive and multiplicative methods in the representation theory (or, also, in order to achieve a better understanding of such an analogy, which to a great extent has to be clarified), we are led to formulate the following conjecture; in its strong version, it transposes to a multiplicative setting the 'if part' of the property of ordinary induction mentioned in the previous paragraph.

Conjecture 4.1. Let G be a group, H a subgroup of G, and D a faithful, quasi-primitive, tensor-indecomposable representation of G. If we have $\bar{D}{\downarrow}_H \simeq \bar{P}_1 \otimes \bar{P}_2$, where P_1 and P_2 are projective representations of H, and $(\deg P_2)^{|G:H|} = \deg D$, then $\bar{D} \simeq \bar{P}_2{\uparrow}^{\otimes G}$ holds (*strong version* of the conjecture) or, at least, there exists a projective representation P of H such that $\bar{D} \simeq \bar{P}{\uparrow}^{\otimes G}$ holds (*weak version* of the conjecture).

Observe that the converse of Conjecture 4.1 (in the strong version) is an easy consequence of the definitions.

In this paper we analyse Conjecture 4.1 assuming solvability for the group G, although the results will be in some cases more general. The following Lemma 4.2 will enable us to exploit, in this analysis, the link between tensor induction and form induction. Recall that the hypotheses of 4.2 determine a situation in which the section $F(G)/Z(G)$ (in the relevant group G) can be thought as a simple $GF(p)[G]$-module, for a suitable prime p, with respect to the conjugation action of G; moreover, that module is endowed with a particular G-invariant nonsingular symplectic $GF(p)$-form (see Lemma 3.5(a)).

Lemma 4.2. *Let G be a group with centre Z and noncentral Fitting subgroup F, and let H be a subgroup of G; also, let D be a faithful, quasi-primitive, tensor-indecomposable representation of G. Assume that we have $\bar{D}{\downarrow}_H \simeq \bar{P}_1 \otimes \bar{P}_2$, where P_1 and P_2 are projective representations of H; if $\deg P_2$ is not 1, and $(\deg P_2)^{|G:H|}$ is a divisor of $\deg D$, then the following conclusions hold:*

(a) *the degree of D is* equal *to* $(\deg P_2)^{|G:H|}$*;*
(b) *denoting by K the kernel of $\bar{P}_1{\downarrow}_F$, we have that K/Z is a submodule of $(F/Z){\downarrow}_H$ which induces F/Z from H. Moreover, the symplectic form on F/Z defined in* 3.5(a) *is nonsingular (that is, it does not vanish) on K/Z.*

Proof. Assume the lemma proved for subgroups which contain Z, and consider the subgroup $M := HZ$. For any given x in M, let h and z be elements, of H and Z respectively, such that $x = hz$; we define the maps $\bar{R}_i : M \to PGL(\deg P_i, \mathbb{C})$ (for i in $\{1,2\}$) by setting $\bar{R}_i(x) := \bar{P}_i(h)$ for all x in M. This is certainly a good definition; moreover, it is easily checked that R_1 and R_2 are projective representations of M, and that $\bar{D}{\downarrow_M} \simeq \bar{R}_1 \otimes \bar{R}_2$ holds. Since $(\deg R_2)^{|G:M|}$ is a divisor of $(\deg R_2)^{|G:H|} = (\deg P_2)^{|G:H|}$, clearly $(\deg R_2)^{|G:M|}$ is a divisor of $\deg D$; now the lemma yields $(\deg R_2)^{|G:M|} = \deg D$, so that $M = H$.

At this stage it is clear that we can assume $H \geq Z$ and, as in Lemma 3.5(b), we see that H is forced to contain F.

Consider now the subgroup $K := \ker(\bar{P}_1{\downarrow_F})$. By Lemma 3.5(a), F/Z is a chief factor of G, hence its order is a power of some prime p; recalling Lemma 2.1(b,c) we see that $|K/Z| = (\deg P_2)^2$, thus $\deg P_2 = p^r$ for a suitable integer r, and $|K/Z| = p^{2r}$. Similarly we have $|F/Z| = (\deg D)^2$ and, since $(\deg P_2)^{|G:H|}$ is a divisor of $\deg D$, we have $\deg D = p^{r|G:H|+w}$ for some integer w, whence $|F/Z| = p^{2(r|G:H|+w)}$. Clearly K/Z is a submodule of $(F/Z){\downarrow_H}$ and, looking at the dimensions of K/Z and F/Z as $GF(p)$-vector spaces, we conclude that $w = 0$ (hence claim (a) is proved) and that K/Z induces F/Z from H. Finally, the relevant symplectic form on F/Z is nonsingular on K/Z, since we have $F = KC_F(K)$ (here we use Lemma 2.1(c,d)), and therefore $F/Z = K/Z \oplus (K/Z)^{\perp}$. ∎

Observe that, in the setting of the previous lemma, the tensor factor P_2 does tensor-induce D provided K/Z form-induces F/Z; indeed, if this is the case, Theorem 3.4 yields a projective representation P of H such that $\bar{D} \simeq \bar{P}{\uparrow^{\otimes G}}$ and $\ker(\bar{P}{\downarrow_F}) = C_F(K)$. Now, we know that any projective representation which tensor-induces D from H is a tensor factor of $D{\downarrow_H}$, that is, there exists a projective representation Y of H such that $\bar{D}{\downarrow_H} \simeq \bar{Y} \otimes \bar{P}$ holds; but we also have $\bar{D}{\downarrow_H} \simeq \bar{P}_1 \otimes \bar{P}_2$ and, since the kernel of $\bar{P}_2{\downarrow_F}$ is also $C_F(K)$ (Lemma 2.1(d)), Theorem 2.2 ensures that P and P_2 are equivalent.

At this stage, it is easily seen that Conjecture 4.2 for groups with noncentral Fitting subgroup is proved (in the strong or in the weak version) if the following statement can be proved (in its strong or weak version, accordingly).

Conjecture 4.3. Let G be a group, H a subgroup of G, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow_H}$. Assume also that V carries a G-invariant nonsingular symplectic $\mathbb{F}$-form f which does not vanish on W. If V is induced by W from H, then V is also form-induced (with respect to f) by W from H (*strong version* of the conjecture) or, at least, V is form-induced (with respect to f) from H, not necessarily by W (*weak version* of the conjecture).

We are now in a position to draw some negative conclusions towards the two conjectures.

5. Some negative answers to the conjectures

Two examples are presented next. The first of them involves a solvable group G which has a faithful, primitive, tensor-indecomposable representation D of degree 5^3, and which has a normal subgroup H of index 3; there exist projective representations P_1 and P_2 of H such that $\bar{D}{\downarrow_H} \simeq \bar{P}_1 \otimes \bar{P}_2$, and such that $\deg P_2 = 5$; nevertheless, D is not tensor-induced by P_2 from H (this is essentially obtained by constructing a counterexample for the strong version of Conjecture 4.3). In other words, Conjecture 4.1 in the strong version is disproved (in the solvable case), thus the analogy between ordinary induction and tensor induction can not be pursued as far as we could initially hope, *even if the subgroup involved is normal of odd prime index.*

The second example, also solvable, shows that both conjectures fail also in the weak versions, in a situation in which the subgroup H has index 2.

Example 5.1. Consider the group

$$E := \langle a, b :\ a^5 = b^5 = [a,b]^5 = 1,\ [a,b,a] = [a,b,b] = 1 \rangle$$

(E is the extraspecial 5-group of order 5^3 and exponent 5). Let σ be the element of $\mathrm{Aut}(E)$ which maps a to b, and b to $b^{-1}a^{-1}$; σ has order 3, and it is the identity on E'. Consider now the automorphism τ of $E \times E \times E$ defined by $(x,y,z)\tau := (z\sigma, x, y)$; if E_0 is the subgroup of $E \times E \times E$ which consists of the elements (x,y,z) in $E' \times E' \times E'$ such that $xyz = 1$, we see that E_0 is invariant under τ. Thus, τ is an automorphism (of order 9) of the extraspecial 5-group $F := (E \times E \times E)/E_0$. If C_9 is a 9-cycle with generator t, we form a semidirect product $G := F \rtimes C_9$ by setting $f^t := f\tau$ for all f in F; G has order $3^2 \cdot 5^7$, and it has a normal subgroup of index 3, namely $H := F \rtimes \langle t^3 \rangle$.

Choose next an irreducible character φ of F with $\varphi(1) \neq 1$ (hence φ is faithful of degree 5^3); since τ centralizes $Z(F)$, the inertia subgroup of φ in G is the whole G (see [**8**, 7.5]). This, together with the fact that G/F is cyclic, ensures ([**8**, 22.3]) that there exists an irreducible character χ of G whose restriction to F is φ. Also, F is the Fitting subgroup of G, whence χ is faithful and we have $Z(F) = Z(H) = Z(G)$. Denoting $Z(F)$ by Z, we observe that F/Z is a simple $GF(5)[G]$-module. Finally, χ is primitive, as G does not have any proper subgroup whose index is a divisor of 5^3. At this stage Theorem 2.2 yields that, denoting by D a representation of G affording χ, D is tensor-indecomposable.

Consider now the simple $GF(5)[G]$-module F/Z. As in Lemma 3.5(a), we define a G-invariant nonsingular symplectic form on it by taking commutators in F. Denoting by y_1 the right Z-coset of $E_0(a,1,1)$, and by y_2 the right Z-coset of $E_0(b,1,1)$, the subspace $Y := \langle y_1, y_2 \rangle$ is indeed an anisotropic simple submodule of $(F/Z){\downarrow_H}$ (that is, the relevant form does not vanish on it), and we have $F/Z = Y \perp Y^t \perp Y^{t^2}$. With respect to the basis $\{y_1, y_2, y_1^t, y_2^t, y_1^{t^2}, y_2^{t^2}\}$, the form is given by a block-scalar matrix, each diagonal block being a hyperbolic plane. If now W is the subspace spanned by $(0,-1,0,-1,0,-1)$ and $(1,1,1,1,1,1)$, it is clear that W is an anisotropic simple submodule of $(F/Z){\downarrow_H}$, such that W^t *is* *not contained* in $W^{\perp}$; in other words, W induces F/Z without form-inducing it (using the results of the following sections it can be shown that, among the 525 anisotropic simple submodules of $(F/Z){\downarrow_H}$, only 21 form-induce F/Z).

We can now achieve the conclusion: let L be the subgroup of G such that $L/Z = W$; we clearly have $F = LC_F(L)$, thus $Z(L)$ coincides with Z, and Theorem 2.2 yields that there exist projective representations P_1 and P_2 of H such that $\bar{D}{\downarrow_H} \simeq \bar{P}_1 \otimes \bar{P}_2$ and $L = \ker(\bar{P}_1{\downarrow_F})$. Moreover, we have $(\deg P_2)^2 = |L/Z|$, hence P_2 has degree 5; but if D were tensor-induced by P_2 then, by Theorem 3.7, $L = C_F(\ker \bar{P}_2{\downarrow_F})$ would provide (by taking L/Z) a submodule of $(F/Z){\downarrow_H}$ which form-induces F/Z. As observed, this is definitely not the case.

Example 5.2. Consider the group E as in the previous example. In $\mathrm{Aut}(E)$ there exist an element i which maps a to b^{-1} and b to a, and an element j which maps a to a^2 and b to b^{-2}; it is easy to check that i and j generate in $\mathrm{Aut}(E)$ a subgroup K which is isomorphic to the quaternion group of order 8, and which centralizes E'. Let now C_2 be a 2-cycle with generator x; denoting by M the semidirect product $E \rtimes K$, we have that $E \times E$ is a normal subgroup of $M \wr C_2$. Moreover, if E_0 is the subgroup of $E \times E$ consisting of the elements (q,r) in $E' \times E'$ such that $qr = 1$, then E_0 is also normal in $M \wr C_2$. In particular, $K \wr C_2$ acts as a group of automorphisms on the extraspecial 5-group $F := (E \times E)/E_0$; also, it is straightforward that $Z(F)$ is centralized by $K \wr C_2$. Consider now the elements $t := (i,1)x$ and $s := (j,j)x$ in $K \wr C_2$; the subgroup Q of $K \wr C_2$ generated by t and s is isomorphic to the quaternion group of order 16. We are finally in a position to define the group $G := F \rtimes Q$, and its subgroup $H := F \rtimes \langle t^2, s\rangle$ (observe that $\langle t^2, s\rangle$ is isomorphic to the quaternion group of order 8).

Consider next an irreducible character φ of F such that $\varphi(1) \neq 1$. As in 5.1, φ is faithful of degree 5^2, its inertia subgroup in G is all of G and, since we have $(|F|, |G:F|) = 1$, there exists an irreducible character χ of G whose restriction to F is φ (see [**8**, 22.3]). Also, F is the Fitting subgroup of G, hence χ is faithful and $Z(F) = Z(H) = Z(G)$. Denoting by Z the centre of F, F/Z is a simple $GF(5)[G]$-module; moreover, G does not have any proper subgroup whose index divides 5^2, so that χ is primitive and the representation D of G which affords χ is tensor-indecomposable.

The simple $GF(5)[G]$-module F/Z is endowed with the usual G-invariant non-singular symplectic form which arises by taking commutators in F; if we choose, as a basis for F/Z, the right Z-cosets of $E_0(a,1)$, $E_0(b,1)$, $E_0(1,a)$ and $E_0(1,b)$ (let us denote them by v_1, v_2, v_3 and v_4), we see that the form is given by a block-scalar matrix having hyperbolic planes as diagonal blocks. Moreover, t maps v_1 to $-v_4$, v_2 to v_3, v_3 to v_1 and v_4 to v_2, whereas s maps v_1 to $2v_3$, v_2 to $-2v_4$, v_3 to $2v_1$ and v_4 to $-2v_2$. It is now easy to check that the subspace W of F/Z spanned by $(1,0,1,0)$ and $(0,1,0,1)$ is an anisotropic simple submodule of $(F/Z){\downarrow_H}$, such that W^t is not contained in $W^{\perp}$. If $(F/Z){\downarrow_H}$ were inhomogeneous, its only proper nonzero submodules would be the two homogeneous components, so we would be forced to have $W^t = W^{\perp}$. Thus $(F/Z){\downarrow_H}$ is the direct sum $W \oplus W^t$

of two isomorphic simple submodules. Since $|\mathrm{End}_H(W)| = 5$, $(F/Z){\downarrow_H}$ has precisely six simple submodules. Among them, we find two isotropic submodules (one spanned by $(1,2,2,1)$ and $(2,-1,1,-2)$, the other the image of this under t). It is then clear that $(W^{\perp})^t = (W^t)^{\perp}$, hence the pair $(W^{\perp}, (W^{\perp})^t)$ does not yield an orthogonal direct decomposition of F/Z as well. In other words, F/Z *is not form-induced* by any submodule of $(F/Z){\downarrow_H}$, thus proving that Conjecture 4.3 fails if the index of H is not assumed to be odd.

In order to show that an odd-index assumption is needed also for Conjecture 4.1, let us consider the subgroup L of G such that $L/Z = W$; exactly the same argument applied in Example 5.1 shows that there exist projective representations P_1 and P_2 of H such that $\bar{D}{\downarrow_H} \simeq \bar{P}_1 \otimes \bar{P}_2$ and $L = \ker(\bar{P}_1{\downarrow_F})$, where P_1 and P_2 have degree 5. Nevertheless, D is not tensor-induced by any projective representation of H; otherwise, by Theorem 3.7, F/Z would be form-induced from H.

In view of the previous examples, it remains to be understood whether *the weak versions* of Conjectures 4.1 and 4.3 are true with an additional odd-index assumption for the subgroup H. The basic idea of the present approach to this problem is to consider, in the setting of Conjecture 4.3, *the whole set* of G-invariant nonsingular symplectic $\mathbb{F}$-forms which can be defined on the module V. For this reason, a discussion on modules endowed with bilinear forms is carried out through the next two sections; it is worth stressing that here, unlike in part of the literature, unitary forms are not regarded as bilinear forms.

6. Modules and bilinear forms: some preliminary remarks

Remark 6.1. Let $\mathbb{F}$ be a field, V a finite dimensional vector space over $\mathbb{F}$, and $f : V \times V \to \mathbb{F}$ a nonsingular bilinear form. It is well known that the map $\psi_f : V \to V^*$ defined by $u(v\psi_f) := f(u,v)$ for all u, v in V is an isomorphism of vector spaces. Consider now an element ε in $\mathrm{End}_{\mathbb{F}}(V)$. Given a v in V, the composite map $\varepsilon(v\psi_f)$ is in V^*, therefore there exists a unique v' in V such that $\varepsilon(v\psi_f) = v'\psi_f$ (that is, $f(u\varepsilon, v) = f(u, v')$ for all u in V). In this way a new map ε', which is in $\mathrm{End}_{\mathbb{F}}(V)$, can be defined by setting $v\varepsilon' := v'$ for all v in V. Moreover, the map τ_f which associates each ε in $\mathrm{End}_{\mathbb{F}}(V)$ to the ε' so defined is an antiautomorphism of the algebra $\mathrm{End}_{\mathbb{F}}(V)$.

Next, let g be another nonsingular bilinear $\mathbb{F}$-form on V; for any given v in V the map $v\psi_g$ is in V^*, hence there exists a unique v' in V such that $v\psi_g = v'\psi_f$, which means $g(u,v) = f(u,v')$ for all u in V. The map γ, defined by setting $v\gamma := v'$ for all v in V, is in $\mathrm{End}_{\mathbb{F}}(V)$, but γ is indeed in $\mathrm{Aut}_{\mathbb{F}}(V)$ as g is nonsingular. Conversely, chosen γ in $\mathrm{Aut}_{\mathbb{F}}(V)$, the map $g : V \times V \to \mathbb{F}$ defined by $g(u,v) := f(u, v\gamma)$ for all u, v in V is a nonsingular bilinear form. In conclusion, a bijection between the set of nonsingular bilinear $\mathbb{F}$-forms on V and $\mathrm{Aut}_{\mathbb{F}}(V)$ arises by means of f. If the form g corresponds to γ in this bijection, and if τ_g is the antiautomorphism of $\mathrm{End}_{\mathbb{F}}(V)$ attached to g, then τ_g and τ_f are linked by the relation $\tau_g = \tau_f \mathrm{Inn}(\gamma^{-1})$, where $\mathrm{Inn}(\gamma^{-1}) : \mathrm{End}_{\mathbb{F}}(V) \to \mathrm{End}_{\mathbb{F}}(V)$ maps ε to $\gamma\varepsilon\gamma^{-1}$.

Let now G be a group, V an $\mathbb{F}G$-module, and $f : V \times V \to \mathbb{F}$ a nonsingular bilinear form which is also G-invariant. In this richer context, the vector space isomorphism ψ_f defined above is indeed an isomorphism of $\mathbb{F}G$-modules between V and its contragredient module V^*. Moreover, consider the correspondence between the set of nonsingular bilinear $\mathbb{F}$-forms on V and $\mathrm{Aut}_{\mathbb{F}}(V)$ which is determined by f; the subset of nonsingular bilinear $\mathbb{F}$-forms on V *which are also G-invariant* is now bijective with $\mathrm{Aut}_{\mathbb{F}G}(V)$.

Assume finally that the module V is simple and the field $\mathbb{F}$ is finite. In this case the restriction of τ_f to $\mathrm{End}_{\mathbb{F}G}(V)$ is a field automorphism and, for any other G-invariant nonsingular bilinear $\mathbb{F}$-form g on V, of course τ_g agrees with τ_f on $\mathrm{End}_{\mathbb{F}G}(V)$. In such a context, we shall denote by τ (or at times, for the sake of emphasis, by τ_V) this field automorphism, dropping any reference to a distinguished form.

Remark 6.2. Let G be a group, V a simple G-module over a finite field $\mathbb{F}$, and f a G-invariant nonsingular bilinear $\mathbb{F}$-form on V which is symmetric or symplectic. Assume also that the characteristic of $\mathbb{F}$ is odd. If g is a G-invariant nonsingular bilinear $\mathbb{F}$-form on V, then g is of the same type as f (that is, symmetric or symplectic) if and only if the element γ of $\mathrm{Aut}_{\mathbb{F}G}(V)$, associated to g in the bijection defined by f, is such that $\gamma^\tau = \gamma$. Assume now that the field $\mathbb{F}$ has characteristic 2: in this case it still holds that, if f is symmetric, then g is also symmetric if and only if γ is fixed by τ.

It is fairly obvious that, as soon as a simple $\mathbb{F}G$-module is self-contragredient (that is, it carries G-invariant nonsingular bilinear $\mathbb{F}$-forms), it is possible to define on it a G-invariant nonsingular bilinear $\mathbb{F}$-form *which is symmetric or symplectic.* In view of that, we can easily prove the next lemma.

Lemma 6.3. *Let G be a group, $\mathbb{F}$ a finite field, and V a simple $\mathbb{F}G$-module which carries G-invariant nonsingular symplectic $\mathbb{F}$-forms. Then the order of the field automorphism τ (as defined in* 6.1*) is at most* 2*.*

Proof. Let f be a G-invariant nonsingular $\mathbb{F}$-form on V which is symmetric or symplectic. If n takes value 1 or -1 in $\mathbb{F}$ according to the symmetric or symplectic nature of f, we have

$$f(x, y\varepsilon^{\tau^2}) = f(x\varepsilon^\tau, y) = nf(y, x\varepsilon^\tau) = nf(y\varepsilon, x) = f(x, y\varepsilon)$$

for all x, y in V and ε in $\mathrm{End}_{\mathbb{F}G}(V)$. The claim follows now from the fact that f is nonsingular. ∎

Before proving Theorem 6.5, which clarifies the role played by the automorphism τ, we need to fix some notations. Let G be a group, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module, and $\mathbb{E}$ the endomorphism ring $\mathrm{End}_{\mathbb{F}G}(V)$. We denote by $V_{\mathbb{E}}$ the simple $\mathbb{E}G$-module which arises regarding V as a vector space over $\mathbb{E}$ (consider here the natural action of $\mathbb{E}$ on V), and by $V^{\mathbb{E}}$ the $\mathbb{E}G$-module $V \otimes_{\mathbb{F}} \mathbb{E}$ (see [**9**, VII,11.1]). Also, if the field $\mathbb{K}$ is a Galois extension of $\mathbb{F}$, U is a $\mathbb{K}G$-module, and η is an element of the Galois group $Gal(\mathbb{K}|\mathbb{F})$, then we denote by U^η the

Galois-conjugate of U by means of η (recall that the underlying set of U^η is the same as of U, and the action of G is unchanged as well; but the action of $\mathbb{K}$ is 'twisted' by η, being defined by $u^k := uk^{\eta^{-1}}$ for all u in U^η and k in K. See [**9**, VII, 1.13] for further details).

The next lemma provides a link between the concepts introduced above.

Lemma 6.4. *Let G be a group, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module, and $\mathbb{E}$ the endomorphism ring of V. Then we have $V^{\mathbb{E}} \simeq \bigoplus_{\eta \in Gal(\mathbb{E}|\mathbb{F})} (V_{\mathbb{E}})^\eta$, where the direct summands are pairwise nonisomorphic simple $\mathbb{E}G$-modules.*

Proof. This follows immediately from [**9**, VII, 1.15 and 1.16a)]. ∎

Theorem 6.5. *Let G be a group, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module with endomorphism ring $\mathbb{E}$, and f a G-invariant nonsingular bilinear $\mathbb{F}$-form on V. Then f yields an isomorphism of $\mathbb{E}G$-modules between $(V_{\mathbb{E}})^\tau$ and $(V_{\mathbb{E}})^*$.*

Proof. Consider the $\mathbb{F}$-vector space V^*, and define an action of $\mathbb{E}$ on it by setting $v\delta^\varepsilon := (v\varepsilon)\delta$ for all v in V, δ in V^* and ε in $\mathbb{E}$; in this way we obtain an $\mathbb{E}$-vector space which we denote by $(V^*)_{\mathbb{E}}$ (this fits the convention established above, since $\mathrm{End}_{\mathbb{F}G}(V^*)$ is $\mathbb{E}$, and the natural action of it on V^* is exactly the one just described). Next, recall that $v^\varepsilon := v\varepsilon^{\tau^{-1}} = v\varepsilon^\tau$ for all v in $(V_{\mathbb{E}})^\tau$ and ε in $\mathbb{E}$.

Consider now the $\mathbb{F}G$-isomorphism $\psi_f : V \to V^*$ defined in 6.1: we claim that ψ_f provides also an $\mathbb{E}G$-isomorphism from $(V_{\mathbb{E}})^\tau$ and $(V^*)_{\mathbb{E}}$. Indeed, for all x, v in V and ε in $\mathbb{E}$, we get

$$x(v^\varepsilon\psi_f) = f(x, v^\varepsilon) = f(x, v\varepsilon^\tau) = f(x\varepsilon, v) = (x\varepsilon)(v\psi_f) = x(v\psi_f)^\varepsilon,$$

hence ψ_f is an isomorphism of $\mathbb{E}$-vector spaces. Moreover, recalling that the relevant structure of $\mathbb{E}G$-module on $(V^*)_{\mathbb{E}}$ is defined by $x(\delta^g) := (x^{g^{-1}})\delta$ for all x in V, δ in $(V^*)_{\mathbb{E}}$ and g in G, it is easily checked that ψ_f is actually an isomorphism of $\mathbb{E}G$-modules.

The final step is to show that $(V^*)_{\mathbb{E}}$ is isomorphic, as an $\mathbb{E}G$-module, to $(V_{\mathbb{E}})^*$. For this purpose, choose a nonzero $\mathbb{F}$-linear map μ from $\mathbb{E}$ to $\mathbb{F}$, and define the map $\beta : (V_{\mathbb{E}})^* \to (V^*)_{\mathbb{E}}$ by $v(\varphi\beta) := (v\varphi)\mu$ for all v in V and φ in $(V_{\mathbb{E}})^*$. It is routine to check that β is an $\mathbb{E}G$-homomorphism. Also, if φ is a nonzero element of $(V_{\mathbb{E}})^*$, then its image is $\mathbb{E}$; now the image of $\varphi\beta$ is $\mathbb{F}$, so that $\varphi\beta$ is not zero as well. This proves that β is actually an isomorphism. ∎

The next theorem clarifies how the nature of the automorphism τ is reflected by the module structure of $V_{\mathbb{E}}$ and $V^{\mathbb{E}}$. Before stating it, we need the following observation: let f be a bilinear $\mathbb{F}$-form on V, and let A be the matrix associated to f with respect to a given $\mathbb{F}$-basis $\{v_1, \ldots, v_n\}$. Now, $\mathcal{B} := \{v_1 \otimes 1, \ldots, v_n \otimes 1\}$ is an $\mathbb{E}$-basis for $V^{\mathbb{E}}$ and A can be regarded as a matrix with entries in $\mathbb{E}$, to which a bilinear $\mathbb{E}$-form $\bar{f}$ is associated (with respect to $\mathcal{B}$). We refer to $\bar{f}$ as to the $\mathbb{E}$*-linear extension of* f; of course, properties like G-invariance or nonsingularity are inherited by $\bar{f}$ if they hold for f.

Theorem 6.6. *Let G be a group, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module with endomorphism ring $\mathbb{E}$, and f a G-invariant nonsingular bilinear $\mathbb{F}$-form on V. Let $\bar{f}$ be the $\mathbb{E}$-form on $V^{\mathbb{E}}$ which arises as the $\mathbb{E}$-linear extension of f. Then the following conditions are equivalent:*

(a) *τ is the identity on $\mathbb{E}$,*
(b) *$V_{\mathbb{E}}$ is self-contragredient,*
(c) *$\bar{f}$ does not vanish on any simple submodule of $V^{\mathbb{E}}$, and any two distinct simple submodules are orthogonal with respect to $\bar{f}$.*

If condition (a) *does not hold, then $V^{\mathbb{E}}$ has a direct decomposition such that the restriction of $\bar{f}$ to any direct summand is nonsingular; the direct summands are pairwise orthogonal with respect to $\bar{f}$, and each of them is the direct sum of two simple submodules on which $\bar{f}$ vanishes and which are contragredients of each other.*

Proof. By Theorem 6.5, we have that (a) implies (b). Since τ is an element of $Gal(\mathbb{E}|\mathbb{F})$, and the Galois-conjugates of $V_{\mathbb{E}}$ are pairwise nonisomorphic (Lemma 6.4), Theorem 6.5 also yields that (b) implies (a). Next, if (c) is assumed, then certainly $V_{\mathbb{E}}$ has to be self-contragredient; what is left is then to show that (b) implies (c). For this purpose observe that, if $V_{\mathbb{E}}$ is assumed self-contragredient, then all the other simple constituent of $V^{\mathbb{E}}$ are also self-contragredient, as obviously we have $((V_{\mathbb{E}})^{\eta})^* \simeq ((V_{\mathbb{E}})^*)^{\eta}$ for all η in $Gal(\mathbb{E}|\mathbb{F})$. Suppose that the form $\bar{f}$ vanishes on a simple constituent of $V^{\mathbb{E}}$, say $(V_{\mathbb{E}})^{\eta}$; since $\bar{f}$ is nonsingular on $V^{\mathbb{E}}$, fixed an element v in $(V_{\mathbb{E}})^{\eta}$ there must be an element w lying in another simple constituent of $V^{\mathbb{E}}$, say $(V_{\mathbb{E}})^{\xi}$, such that $\bar{f}(v,w)$ is not zero. But now $\bar{f}$ provides an $\mathbb{E}G$-isomorphism between $(V_{\mathbb{E}})^{\eta}$ and $((V_{\mathbb{E}})^{\xi})^*$ (which is in turn isomorphic to $(V_{\mathbb{E}})^{\xi}$), and this is a contradiction. We conclude that $\bar{f}$ does not vanish on any simple constituent of $V^{\mathbb{E}}$. This also means that the orthogonal (with respect to $\bar{f}$) of a simple constituent is a direct complement for it. On the other hand each simple constituent is a homogeneous component, and therefore it has a unique complement in $V^{\mathbb{E}}$, namely the direct sum of the other homogeneous components. It follows that the orthogonal of a simple constituent contains all the other simple constituents.

We move now to the last claim. First of all, if $V_{\mathbb{E}}$ is not self-contragredient, then the same holds for all the simple constituents of $V^{\mathbb{E}}$ (this follows from the fact that they lie in a single Galois orbit). Now of course $\bar{f}$ has to vanish on all of them, and each has to be orthogonal (with respect to $\bar{f}$) to all the others except to the one that is contragredient to it. We know that the contragredient of $(V_{\mathbb{E}})^{\eta}$ is $(V_{\mathbb{E}})^{\eta\tau}$, so let us bracket in pairs the simple direct summands of $V^{\mathbb{E}}$, matching $(V_{\mathbb{E}})^{\eta}$ with $(V_{\mathbb{E}})^{\eta\tau}$ for all η in $Gal(\mathbb{E}|\mathbb{F})$. We get a direct decomposition of $V^{\mathbb{E}}$ in which each summand is the direct sum of two simple submodules, on which $\bar{f}$ vanishes, and which are contragredients of each other. Moreover, these two-component summands are pairwise orthogonal to each other, and so the restriction of the nonsingular $\bar{f}$ to each of them must be nonsingular. ∎

Remark 6.7. In view of the discussion carried out so far, we can draw the following picture of the relationship between simple modules and bilinear forms. Let G be a group, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module which is self-contragredient, and $\mathbb{E}$ the endomorphism ring of V. Then two different situations may occur.

(a) *If also* $V_{\mathbb{E}}$ *is self-contragredient*, then it follows from 6.2 and 6.6 that *either* the G-invariant nonsingular bilinear forms on both V and $V_{\mathbb{E}}$ are all symmetric, *or* they are all symplectic. This of course makes sense if $\mathbb{F}$ has odd characteristic; otherwise, it will be clear by Proposition 7.2 (together with [**9**, VII, 8.13]) that all the forms are indeed symplectic unless V is the trivial module.

(b) *If* $V_{\mathbb{E}}$ *is not self-contragredient*, then V carries G-invariant bilinear $\mathbb{F}$-forms of both the symmetric and the symplectic type, and also of a third different type. Again we have to treat separately the case in which $\mathbb{F}$ has characteristic 2; in that case, V carries G-invariant nonsingular bilinear $\mathbb{F}$-forms of both the symplectic and the non-symmetric type.

7. Equivalence of forms and induction of forms

In what follows we focus on a particular type of forms (the symplectic ones) for ease of the exposition, keeping in mind that essentially nothing changes in the symmetric context.

Definition 7.1. Let G be a group, $\mathbb{F}$ a finite field, and V a simple $\mathbb{F}G$-module; also, let f, g, h be G-invariant nonsingular symplectic $\mathbb{F}$-forms on V. If there exists ε in $\mathrm{Aut}_{\mathbb{F}G}(V)$ such that $g(u,v) = f(u\varepsilon, v\varepsilon)$ holds for all u and v in V, then we say that g *is equivalent to* f (and we write $g \sim f$) *by means of* ε. This defines an equivalence relation on the set of G-invariant nonsingular symplectic $\mathbb{F}$-forms on V, and we have $g \sim f$ if and only if the element γ_g of $\mathrm{Aut}_{\mathbb{F}G}(V)$, which corresponds to g in the bijection yielded by f (see 6.1), is equal to $\varepsilon\varepsilon^{\tau}$ for some ε in $\mathrm{Aut}_{\mathbb{F}G}(V)$. If f is chosen as a distinguished form, then we have $g \sim h$ if and only if $\gamma_g\gamma_h^{-1}$ is an element of the subgroup K of $\mathrm{Aut}_{\mathbb{F}G}(V)$ defined by $K := \{\varepsilon\varepsilon^{\tau} \,:\, \varepsilon \in \mathrm{Aut}_{\mathbb{F}G}(V)\}$. Of course the equivalence relation is not affected by the choice of f.

Assume now the setting of 7.1; if p is the characteristic of $\mathbb{F}$, then the order of the field $\mathbb{E} = \mathrm{End}_{\mathbb{F}G}(V)$ is p^{α} for some integer α, and $\mathrm{Aut}(\mathbb{E})$ is a cyclic group of order α whose elements are the p^{i}-th powering maps on $\mathbb{E}$ (i is an integer running from 0 to $\alpha - 1$). Suppose that τ is not the identity on $\mathbb{E}$; in this case α is necessarily an even number, say 2β, and τ is given by the p^{β}-th powering. Using that $\mathrm{Aut}_{\mathbb{F}G}(V)$ is a cyclic group of order $p^{2\beta} - 1$, we can conclude that *if* τ *is not the identity on* $\mathbb{E}$, *then all the elements of* $\mathrm{Aut}_{\mathbb{F}G}(V)$ *which are fixed by* τ *are equal to* $\varepsilon\varepsilon^{\tau}$ *for some* ε *in* $\mathrm{Aut}_{\mathbb{F}G}(V)$. *Obviously, if* τ *is the identity on* $\mathbb{E}$, *then the elements of* $\mathrm{Aut}_{\mathbb{F}G}(V)$ *which are (fixed by* τ *and) equal to* $\varepsilon\varepsilon^{\tau}$ *for some* ε *in* $\mathrm{Aut}_{\mathbb{F}G}(V)$ *are precisely the squares of* $\mathrm{Aut}_{\mathbb{F}G}(V)$.

From the previous discussion, we derive at once the following proposition.

Proposition 7.2. Let G be a group, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module which carries a G-invariant nonsingular symplectic $\mathbb{F}$-form.

(a) If τ is not the identity on $\mathrm{End}_{\mathbb{F}G}(V)$, then there is a unique equivalence class of G-invariant nonsingular symplectic $\mathbb{F}$-forms on V. This also holds if τ is the identity on $\mathrm{End}_{\mathbb{F}G}(V)$, provided the characteristic of $\mathbb{F}$ is 2.

(b) If τ is the identity on $\mathrm{End}_{\mathbb{F}G}(V)$, and $\mathbb{F}$ has odd characteristic, then there are two equivalence classes of G-invariant nonsingular symplectic $\mathbb{F}$-forms on V; if f and g are two such forms, and g is attached to the element γ of $\mathrm{Aut}_{\mathbb{F}G}(V)$ through the bijection defined by f, then f and g are equivalent if and only if γ is a square in $\mathrm{Aut}_{\mathbb{F}G}(V)$.

We define next the concept of *induced form.* Henceforth, through this section, we shall be assuming that *G is a group, H is a subgroup of G whose index is n, $\mathbb{F}$ is a finite field, V is a simple $\mathbb{F}G$-module, and W is a submodule of $V{\downarrow_H}$ such that $V \simeq W{\uparrow^G}$.* In this setting, given an element δ of $\mathrm{End}_{\mathbb{F}H}(W)$, we denote by $\bar{\delta}$ the unique element of $\mathrm{End}_{\mathbb{F}G}(V)$ whose restriction to W is δ (recall that, if $\{g_1, \dots, g_n\}$ is a right transversal for H in G, then $\bar{\delta}$ is defined by $v\bar{\delta} := \sum_{j=1}^{n}(v_j\delta)^{g_j}$ for all v in V, where $(v_1, \dots, v_n)$ is the uniquely determined sequence in W such that $v = \sum_{j=1}^{n} v_j^{g_j}$). The map which associates every δ in $\mathrm{End}_{\mathbb{F}H}(W)$ to $\bar{\delta}$ is a monomorphism of fields; in the sequel, we shall make no distinction (by means of notation) between the abstract field $\mathrm{End}_{\mathbb{F}H}(W)$ and the mentioned copy of it embedded in $\mathrm{End}_{\mathbb{F}G}(V)$.

Definition 7.3. Let f be an H-invariant nonsingular symplectic $\mathbb{F}$-form on W, and let $\{g_1, \dots, g_n\}$ be a right transversal for H in G. Given u and v in V, consider the uniquely determined sequences $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ of elements in W such that $u = \sum_{j=1}^{n} u_j^{g_j}$ and $v = \sum_{j=1}^{n} v_j^{g_j}$. We define the map $f{\uparrow^V}\colon V \times V \to \mathbb{F}$ by setting

$$f{\uparrow^V}(u, v) := \sum_{j=1}^{n} f(u_j, v_j).$$

It is easily checked that $f{\uparrow^V}$ is a G-invariant nonsingular symplectic $\mathbb{F}$-form on V. Observe that, although the construction above involves the choice of a transversal for H in G, the result of this construction is not affected at all by such a choice, so that we can safely refer to $f{\uparrow^V}$ as to the form on V which is *induced by f from W.* Also, it is clear that the restriction of $f{\uparrow^V}$ to W is f.

The following remark points out that the process of induction is 'well defined' with respect to the equivalence relation of 7.1.

Remark 7.4. Let f and g be H-invariant nonsingular symplectic $\mathbb{F}$-forms on W; if γ is the element of $\mathrm{Aut}_{\mathbb{F}H}(W)$ such that $g(w_1, w_2) = f(w_1, w_2\gamma)$ for all w_1, w_2 in W, then we have $g{\uparrow^V}(v_1, v_2) = f{\uparrow^V}(v_1, v_2\bar{\gamma})$ for all v_1, v_2 in V. Moreover, if g is equivalent to f by means of an element ε of $\mathrm{Aut}_{\mathbb{F}H}(W)$, then $g{\uparrow^V}$ is equivalent to $f{\uparrow^V}$ by means of $\bar{\varepsilon}$.

It will also be useful to observe that, if W carries H-invariant nonsingular symplectic $\mathbb{F}$-forms (thus, in view of 7.3, V carries G-invariant nonsingular symplectic $\mathbb{F}$-forms), and if the automorphism τ_V is the identity on $\mathrm{End}_{\mathbb{F}G}(V)$, then τ_W is the identity on $\mathrm{End}_{\mathbb{F}H}(W)$ as well. This follows from the fact that $\overline{\delta^{\tau_W}} = \bar{\delta}^{\tau_V}$ for all δ in $\mathrm{End}_{\mathbb{F}H}(W)$.

We are now in a position to change our point of view, outlining a connection between the concept of form induction and the concept of induced forms.

Remark 7.5. In the established setting, assume that the $\mathbb{F}H$-module W carries H-invariant nonsingular symplectic $\mathbb{F}$-forms; then V carries G-invariant nonsingular symplectic $\mathbb{F}$-forms. Chosen a form $\langle\ ,\ \rangle$ among them, we observe what follows.

If there exists an H-invariant nonsingular symplectic $\mathbb{F}$-form f on W such that $\langle\ ,\ \rangle \sim f{\uparrow}^V$, then V is form-induced from H (with respect to the form $\langle\ ,\ \rangle$) by a submodule of $V{\downarrow}_H$ isomorphic to W. More precisely, if ε is an element of $\mathrm{Aut}_{\mathbb{F}G}(V)$ such that $f{\uparrow}^V(u,v) = \langle u\varepsilon, v\varepsilon\rangle$ for all u, v in V, then V is form-induced by $W\varepsilon$ (this is an easy application of definitions).

Conversely, if $\langle\ ,\ \rangle$ is *not* equivalent to any form which is induced from W, then V is not form-induced from H by any $\mathbb{F}H$-submodule isomorphic to W. Indeed, for any submodule Z of $V{\downarrow}_H$ which is isomorphic to W, there exists an element ε in $\mathrm{Aut}_{\mathbb{F}G}(V)$ such that $Z = W\varepsilon$ and, if V is form-induced by $W\varepsilon$, then we can define an H-invariant nonsingular symplectic $\mathbb{F}$-form f on W by setting $f(w_1, w_2) := \langle w_1\varepsilon, w_2\varepsilon\rangle$ for all w_1, w_2 in W. Now we see that the form $f{\uparrow}^V$ is equivalent to $\langle\ ,\ \rangle$ by means of ε: for u and v in V, let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be the uniquely determined sequences in W such that $u = \sum_{j=1}^n u_j^{g_j}$ and $v = \sum_{j=1}^n v_j^{g_j}$ (as usual, $\{g_1, \ldots, g_n\}$ denotes a right transversal for H in G); we get

$$\begin{aligned}\langle u\varepsilon, v\varepsilon\rangle &= \Big\langle(\sum_{j=1}^n u_j^{g_j})\varepsilon, (\sum_{j=1}^n v_j^{g_j})\varepsilon\Big\rangle = \Big\langle \sum_{j=1}^n (u_j\varepsilon)^{g_j}, \sum_{j=1}^n (v_j\varepsilon)^{g_j}\Big\rangle = \sum_{j=1}^n \langle (u_j\varepsilon)^{g_j}, (v_j\varepsilon)^{g_j}\rangle \\ &= \sum_{j=1}^n \langle u_j\varepsilon, v_j\varepsilon\rangle = \sum_{j=1}^n f(u_j, v_j) = f{\uparrow}^V(u,v).\end{aligned}$$

Lemma 7.7 will provide an effective criterion to determine the existence of a 'possibly bad' form on the module V (that is, a G-invariant nonsingular symplectic $\mathbb{F}$-form on V *not equivalent* to any form which is induced from W); we shall see that such a form exists if and only if there are two equivalence classes of forms on V and, in the process of induction, the two classes of forms on W are fused together, so that the other class of forms on V remains uncovered.

Since Proposition 7.2 ensures that, if the field $\mathbb{F}$ has characteristic 2, we have in any case a unique equivalence class of forms, we can safely restrict our attention to the case in which the characteristic of $\mathbb{F}$ is odd.

Before going through Lemma 7.7, we state an easy introductory proposition.

Proposition 7.6. Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be finite fields of odd characteristic, such that $\mathbb{K}_1$ is a subfield of $\mathbb{K}_2$. Then the following conditions are equivalent:

(a) the degree of $\mathbb{K}_2$ over $\mathbb{K}_1$ (as a field extension) is an even number;
(b) there exists an element ξ in $\mathbb{K}_2 \setminus \mathbb{K}_1$ such that ξ^2 lies in $\mathbb{K}_1$;

(c) all the elements of $\mathbb{K}_1$ are squares in $\mathbb{K}_2$.

Lemma 7.7. *Let G be a group, H a subgroup of G, $\mathbb{F}$ a field of odd characteristic, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow_H}$ such that $V \simeq W{\uparrow^G}$. Assume that W carries H-invariant nonsingular symplectic $\mathbb{F}$-forms; then the following conditions are equivalent:*

(a) *the automorphism τ_V is the identity on $\mathrm{End}_{\mathbb{F}G}(V)$, and the degree of $\mathrm{End}_{\mathbb{F}G}(V)$ over $\mathrm{End}_{\mathbb{F}H}(W)$ (as a field extension) is an even number;*

(b) *there exists a G-invariant nonsingular symplectic $\mathbb{F}$-form on V which is not equivalent to any of the forms induced from W.*

Moreover, for any G-invariant nonsingular symplectic $\mathbb{F}$-form on V, there exists a submodule of $V{\downarrow_H}$, isomorphic to W, on which the form under consideration does not vanish.

Proof. Assume condition (a). Since τ_V is the identity on $\mathrm{End}_{\mathbb{F}G}(V)$, Proposition 7.2(b) ensures that the set of G-invariant nonsingular symplectic $\mathbb{F}$-forms on V is partitioned into two equivalence classes, say C_V^1 and C_V^2. As observed, we also have two equivalence classes C_W^1 and C_W^2 of H-invariant nonsingular symplectic $\mathbb{F}$-forms on W. Suppose that an element f of C_W^1 yields an induced form $f{\uparrow^V}$ which lies in C_V^1: Remark 7.4 ensures that any form in C_W^1 induces to C_V^1 as well. Consider now an element g in C_W^2, and the element γ in $\mathrm{Aut}_{\mathbb{F}H}(W)$ such that $g(w_1, w_2) = f(w_1, w_2\gamma)$ holds for all w_1, w_2 in W; we have $g{\uparrow^V}(u,v) = f{\uparrow^V}(u, v\bar{\gamma})$ for all u, v in V and, although γ is not a square in $\mathrm{Aut}_{\mathbb{F}H}(W)$, $\bar{\gamma}$ is indeed a square in $\mathrm{Aut}_{\mathbb{F}G}(V)$ by the previous proposition. This yields that $g{\uparrow^V}$ is equivalent to $f{\uparrow^V}$, so that the process of induction 'maps' also the class C_W^2 to C_V^1. Now, given a form in C_V^2, this can not be equivalent to any of the forms induced from W.

Conversely, let us assume condition (b). Of course we must have two equivalence classes of G-invariant nonsingular symplectic $\mathbb{F}$-forms on V (otherwise just consider a form f on W and induce it up; now all the forms on V are equivalent to $f{\uparrow^V}$), whence τ_V is the identity. Moreover, let f and g be inequivalent forms on W; if γ is the element of $\mathrm{Aut}_{\mathbb{F}H}(W)$ such that $g(w_1, w_2) = f(w_1, w_2\gamma)$ holds for all w_1, w_2 in W, then we know that also $g{\uparrow^V}(u,v) = f{\uparrow^V}(u, v\bar{\gamma})$ holds for all u, v in V. Although γ is not a square in $\mathrm{Aut}_{\mathbb{F}H}(W)$, our assumption forces $\bar{\gamma}$ to become a square in $\mathrm{Aut}_{\mathbb{F}G}(V)$, and now the previous proposition yields that the degree of $\mathrm{End}_{\mathbb{F}G}(V)$ over $\mathrm{End}_{\mathbb{F}H}(W)$ as a field extension in an even number.

We move now to the last part of the statement. First of all, let $\langle\langle\ ,\ \rangle\rangle$ and $[[\ ,\ ]]$ be G-invariant nonsingular symplectic $\mathbb{F}$-forms on V, and assume that X is a submodule of $V{\downarrow_H}$, isomorphic to W, on which $\langle\langle\ ,\ \rangle\rangle$ does not vanish; assume also that $\langle\langle\ ,\ \rangle\rangle$ is equivalent to $[[\ ,\ ]]$ by means of the element ε in $\mathrm{Aut}_{\mathbb{F}G}(V)$. Then it is easy to see that $[[\ ,\ ]]$ does not vanish on $X\varepsilon$. Moreover, any form on V which is induced from W of course does not vanish on W. Therefore, we only have to deal with the case in which there are forms on V not induced from W, and it will be enough to prove the claim for one of them.

For this purpose, consider an H-invariant nonsingular symplectic $\mathbb{F}$-form f on W, and denote by $\langle\ ,\ \rangle$ the form $f{\uparrow}^V$. Choose a generator, ζ say, of $\mathrm{Aut}_{\mathbb{F}G}(V)$; it is clear that ζ is not a square in $\mathrm{Aut}_{\mathbb{F}G}(V)$, hence the form $[\ ,\ ]$ defined by $[u,v] := \langle u, v\zeta\rangle$ for all u,v in V is not equivalent to any form which is induced from W; moreover, $\mathrm{End}_{\mathbb{F}G}(V)$ can be obtained as a simple extension of $\mathrm{End}_{\mathbb{F}H}(W)$ by adjoining ζ. Let us assume that $W\eta\zeta$ is orthogonal, with respect to $\langle\ ,\ \rangle$, to $W\eta$ for all η in $\mathrm{Aut}_{\mathbb{F}G}(V)$; then, in particular, $W\zeta^{r-1}$ is orthogonal to $W\zeta^r$, and $W(1+\zeta^{r-1})$ is orthogonal to $W(1+\zeta^{r-1})\zeta$ for all r in $\{1,\ldots,n := |\,\mathrm{End}_{\mathbb{F}G}(V) : \mathrm{End}_{\mathbb{F}H}(W)|\}$. Therefore we have

$$0 = \langle w + w\zeta^{r-1}, z\zeta + z\zeta^r\rangle = 2\langle w, z\zeta^r\rangle$$

for all w,z in W (recall that in the present situation τ_V is the identity, so that $\langle w\zeta^{r-1}, z\zeta\rangle = \langle w, z\zeta^r\rangle$ holds). It follows that W is orthogonal to $W\zeta^r$ for all r in $\{1,\ldots,n\}$. Now, there exists a sequence $(\delta_0,\ldots,\delta_{n-1})$ of elements in $\mathrm{End}_{\mathbb{F}H}(W)$ such that $\zeta^n = \sum_{j=0}^{n-1}\delta_j\zeta^j$ holds. Hence, for all w,z in W, we get

$$0 = \langle w, z\zeta^n\rangle = \langle w, z\delta_0 + z\delta_1\zeta + \cdots + z\delta_{n-1}\zeta^{n-1}\rangle = \langle w, z\delta_0\rangle,$$

but now δ_0 has to be 0, a contradiction. We conclude that there exists η in $\mathrm{Aut}_{\mathbb{F}G}(V)$ such that $W\eta$ is not orthogonal to $W\eta\zeta$, thus it is clear that $[\ ,\ ]$ does not vanish on $W\eta$, and our claim is proved. ■

8. First positive results for the conjectures

It would be desirable, for our purposes, to argue that the situation outlined in the two equivalent conditions of Lemma 7.7 can not arise at all if we assume that H has odd index in G. Although this is unfortunately not the case (as Example 11.1 will show), we can anyway draw positive conclusions towards Conjectures 4.1 and 4.3 in a particular setting, namely when the subgroup H is assumed to be normal (of odd index). Prior to that, we need to state a lemma (8.1) which can be essentially derived from [**9**, VII, 4.12b)].

Lemma 8.1. *Let G be a group, H a subgroup of G, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow}_H$ such that $V \simeq W{\uparrow}^G$. Let U be the homogeneous component of W in the socle of $V{\downarrow}_H$. Then, denoting by n the degree of $\mathrm{End}_{\mathbb{F}G}(V)$ as a field extension of $\mathrm{End}_{\mathbb{F}H}(W)$, the composition length of U (as an $\mathbb{F}H$-module) is n. Moreover, U is a direct summand in $V{\downarrow}_H$, it has a unique direct complement Y, and Y is such that $\mathrm{Hom}_{\mathbb{F}H}(W,Y) = 0$.*

Theorem 8.2. *Let G be a group, H a normal subgroup of G having odd index, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow}_H$. Assume also that V carries a G-invariant nonsingular symplectic $\mathbb{F}$-form f which does not vanish on W. If V is induced by W from H, then V is also form-induced (with respect to f) from H. Moreover, a submodule of $V{\downarrow}_H$ which form-induces V can be chosen isomorphic to W.*

Proof. If $\mathbb{F}$ has characteristic 2, the result is proved by Proposition 7.2(a), together with Remark 7.5. If $\mathbb{F}$ has odd characteristic, we achieve the conclusion by means of Lemma 7.7 (again followed by 7.5) since, by 8.1, the degree of $\mathrm{End}_{\mathbb{F}G}(V)$ over $\mathrm{End}_{\mathbb{F}H}(W)$ is certainly an odd number (observe that, by Clifford's Theorem, the multiplicity of W as a composition factor of $V{\downarrow_H}$ is odd). ∎

Theorem 8.3. *Let G be a group with noncentral Fitting subgroup, H a normal subgroup of G having odd index, and D a faithful, quasi-primitive, tensor-indecomposable representation of G. Assume that we have $\bar{D}{\downarrow_H} \simeq \bar{P}_1 \otimes \bar{P}_2$, where P_1 and P_2 are projective representations of H. If $\deg P_2$ is not 1, and $(\deg P_2)^{|G:H|}$ is a divisor of $\deg D$, then we have $(\deg P_2)^{|G:H|} = \deg D$, and there exists a projective representation P of H such that $\bar{D} \simeq \bar{P}{\uparrow^{\otimes G}}$ holds.*

Proof. Start by applying Lemma 4.2; then the claim follows from Theorem 8.2 together with Theorem 3.7. ∎

Observe that Theorem 8.3 and Example 5.2 provide a full answer to the problem expressed by Conjecture 4.1 (for groups whose Fitting subgroup is noncentral) if we restrict our attention to normal subgroups. In fact, we conclude that Conjecture 4.1 is true for normal subgroups of odd index, whereas it fails in general when the index of H is even.

9. Induction from maximal subgroups

The main results of this section are Theorem 9.7 and Theorem 9.10. Roughly speaking, they deal with a situation in which a simple $\mathbb{F}G$-module V is induced from a maximal subgroup H of G (of course the setting is in both cases much more specific), and their aim is to achieve some control on the structure of $V{\downarrow_H}$. Such a control, together with the discussion carried out so far, will yield some more evidence for Conjectures 4.1 and 4.3 with the odd index assumption, also for subgroups which are not necessarily normal (see Section 10).

Lemma 9.1. *Let H be a group, L a normal subgroup of H, $\mathbb{F}$ a finite field, and X a 1-dimensional $\mathbb{F}H$-module whose kernel contains L. Let W be a simple $\mathbb{F}H$-module. Then $W \otimes X$ and W have the same (nonzero) multiplicity as composition factors in the socle of $W{\downarrow_L}{\uparrow^H}$.*

Proof. By Nakayama reciprocity ([**9**, VII, 4.10]) we have

$$\mathrm{Hom}_{\mathbb{F}H}(W \otimes X, W{\downarrow_L}{\uparrow^H}) \simeq \mathrm{Hom}_{\mathbb{F}L}((W \otimes X){\downarrow_L}, W{\downarrow_L}) \simeq$$
$$\simeq \mathrm{Hom}_{\mathbb{F}L}(W{\downarrow_L}, W{\downarrow_L}) \simeq \mathrm{Hom}_{\mathbb{F}H}(W, W{\downarrow_L}{\uparrow^H}),$$

where the symbol '$\simeq$' denotes an isomorphism of vector spaces. Moreover, since X is 1-dimensional, it is not hard to check that $\mathrm{End}_{\mathbb{F}H}(W \otimes X)$ is isomorphic, as a vector space, to $\mathrm{End}_{\mathbb{F}H}(W)$. Denoting by $m(Z)$ the multiplicity of a simple $\mathbb{F}H$-module Z as a composition factor in the socle of $W{\downarrow_L}{\uparrow^H}$, by [**9**, VII, 4.12b)] we get

$$m(W\otimes X)=\frac{\dim_{\mathbb{F}}\mathrm{Hom}_{\mathbb{F}H}(W\otimes X, W{\downarrow_L}{\uparrow^H})}{\dim_{\mathbb{F}}\mathrm{End}_{\mathbb{F}H}(W\otimes X)}=$$
$$=\frac{\dim_{\mathbb{F}}\mathrm{Hom}_{\mathbb{F}H}(W, W{\downarrow_L}{\uparrow^H})}{\dim_{\mathbb{F}}\mathrm{End}_{\mathbb{F}H}(W)}=m(W),$$

as desired. ∎

The well known facts recalled (without proofs) in Propositions 9.2 and 9.3 will be useful for the critical Lemma 9.4. As in Section 6, given an $\mathbb{F}G$-module U and a field extension $\mathbb{K}$ of $\mathbb{F}$, we will denote by $U^{\mathbb{K}}$ the $\mathbb{K}G$-module $U\otimes_{\mathbb{F}}\mathbb{K}$.

Proposition 9.2. Let G be a group, $\mathbb{F}$ a field, U and V $\mathbb{F}G$-modules, and $\mathbb{K}$ a field extension of $\mathbb{F}$; then $\mathrm{Hom}_{\mathbb{K}G}(U^{\mathbb{K}},V^{\mathbb{K}})$ and $\mathrm{Hom}_{\mathbb{F}G}(U,V)\otimes_{\mathbb{F}}\mathbb{K}$ are isomorphic as vector spaces over $\mathbb{K}$. In particular, $\dim_{\mathbb{K}}\mathrm{Hom}_{\mathbb{K}G}(U^{\mathbb{K}},V^{\mathbb{K}})=\dim_{\mathbb{F}}\mathrm{Hom}_{\mathbb{F}G}(U,V)$.

Proof. See [**7**, V, 11.9]. ∎

Proposition 9.3. Let G be a group, H a subgroup of G, $\mathbb{F}$ a field, and $\mathbb{K}$ a field extension of $\mathbb{F}$; also, let U, V, Z be $\mathbb{F}G$-modules, and W an $\mathbb{F}H$-module. Then the following properties hold:

(a) $U\otimes(V\oplus Z)\simeq(U\otimes V)\oplus(U\otimes Z)$;
(b) $(U\oplus V)^{\mathbb{K}}\simeq U^{\mathbb{K}}\oplus V^{\mathbb{K}}$;
(c) $(V{\downarrow_H})^{\mathbb{K}}\simeq(V^{\mathbb{K}}{\downarrow_H})$;
(d) $(W{\uparrow^G})^{\mathbb{K}}\simeq(W^{\mathbb{K}}){\uparrow^G}$.

Lemma 9.4. *Let H be a group, L a normal subgroup of H, $\mathbb{F}$ a finite field of odd characteristic, and W an absolutely simple $\mathbb{F}H$-module. Assume that there exists an $\mathbb{F}H$-module X such that its kernel M contains L, $|H:M|=2$, and $W\otimes X$ is isomorphic to W. Then the multiplicity of W as a composition factor in the socle of $W{\downarrow_L}{\uparrow^H}$ is a positive even number.*

Proof. As the first step, we prove that if there exists a finite degree field extension $\mathbb{K}$ of $\mathbb{F}$ such that the lemma holds with $\mathbb{K}$ in place of $\mathbb{F}$, then the lemma holds for $\mathbb{F}$ as well. Let $\mathbb{K}$ be such an extension, and consider the module $W^{\mathbb{K}}$. This is simple (as W is absolutely simple), but $W^{\mathbb{K}}$ is also absolutely simple because (by Proposition 9.2) we have $\dim_{\mathbb{K}}\mathrm{End}_{\mathbb{K}H}(W^{\mathbb{K}})=\dim_{\mathbb{F}}\mathrm{End}_{\mathbb{F}H}(W)=1$. Now, the regular module $\mathbb{F}[H/M]$ is the direct sum of X and the 1-dimensional trivial $\mathbb{F}H$-module. By 9.3(a), and our assumption $W\simeq W\otimes X$, we get

$$W{\downarrow_M}{\uparrow^H}\simeq W\otimes\mathbb{F}[H/M]\simeq W\oplus W$$

(here we also used [**9**, VII, 4.15]), and therefore, by 9.3(b,c,d),

$$(W^{\mathbb{K}}){\downarrow_M}{\uparrow^H}\simeq(W{\downarrow_M}{\uparrow^H})^{\mathbb{K}}\simeq(W\oplus W)^{\mathbb{K}}\simeq W^{\mathbb{K}}\oplus W^{\mathbb{K}}.$$

On the other hand we have $(W^{\mathbb{K}}){\downarrow_M}{\uparrow^H}\simeq W^{\mathbb{K}}\otimes\mathbb{K}[H/M]$, and of course $\mathbb{K}[H/M]$ is the direct sum of the 1-dimensional trivial $\mathbb{K}H$-module and another 1-dimensional $\mathbb{K}H$-module, say X' (which is indeed $X^{\mathbb{K}}$), whose kernel is M. We conclude that

$W^{\mathbb{K}} \otimes X'$ is forced to be isomorphic to $W^{\mathbb{K}}$. Now, as we are assuming that the lemma holds for $\mathbb{K}$, we have that the multiplicity of $W^{\mathbb{K}}$ in the socle of $(W^{\mathbb{K}}){\downarrow_L}{\uparrow^H}$ is a positive even number. But, as we observed in proving Lemma 9.1, this multiplicity is given by $\dim_{\mathbb{K}} \operatorname{Hom}_{\mathbb{K}H}(W^{\mathbb{K}}, (W^{\mathbb{K}}){\downarrow_L}{\uparrow^H})$, and we have

$$\begin{aligned}\dim_{\mathbb{K}} \operatorname{Hom}_{\mathbb{K}H}(W^{\mathbb{K}}, (W^{\mathbb{K}}){\downarrow_L}{\uparrow^H}) &= \dim_{\mathbb{K}} \operatorname{Hom}_{\mathbb{K}H}(W^{\mathbb{K}}, (W{\downarrow_L}{\uparrow^H})^{\mathbb{K}}) \\ &= \dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}H}(W, W{\downarrow_L}{\uparrow^H}).\end{aligned}$$

The claim is now proved, as the last member in the chain of equalities above is the multiplicity of W in the socle of $W{\downarrow_L}{\uparrow^H}$.

Since there exists a finite degree field extension of $\mathbb{F}$ which is a splitting field for G and for all its subgroups (see [**9**, VII, 2.6]), by the previous step we can certainly assume that $\mathbb{F}$ itself is such a splitting field.

Observe that, as the multiplicity of W in $W{\downarrow_M}{\uparrow^H}$ is 2, we get

$$\dim_{\mathbb{F}} \operatorname{End}_{\mathbb{F}M}(W{\downarrow_M}) = \dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}H}(W, W{\downarrow_M}{\uparrow^H}) = 2.$$

Now, $W{\downarrow_M}$ is certainly not simple (because $\mathbb{F}$ is a splitting field for M), and it is not homogeneous with composition length 2 (in that case $\dim_{\mathbb{F}} \operatorname{End}_{\mathbb{F}M}(W{\downarrow_M})$ would be 4). The only other possibility is $W{\downarrow_M} = Y \oplus Y^h$, where Y is a simple $\mathbb{F}M$-module, h is in $H \setminus M$, and Y^h is not isomorphic to Y. This yields that the composition length of $W{\downarrow_L}$ is an even number, since it is twice the composition length of $Y{\downarrow_L}$. On the other hand, assume that $W{\downarrow_L}$ has k homogeneous components, each having composition length m: the composition length of $W{\downarrow_L}$ is now given by km, and we know that this is an even number. But $\operatorname{End}_{\mathbb{F}L}(W{\downarrow_L})$ is isomorphic to the direct sum of k copies of $\operatorname{Mat}(m, \mathbb{F})$, hence its dimension over $\mathbb{F}$ is km^2, an even number as well. Since this is also the dimension over $\mathbb{F}$ of $\operatorname{Hom}_{\mathbb{F}H}(W, W{\downarrow_L}{\uparrow^H})$, which is in turn the multiplicity of W in the socle of $W{\downarrow_L}{\uparrow^H}$, the proof is complete. ∎

Consider now the following situation: G is a solvable group, H a subgroup of G having odd prime index, $\mathbb{F}$ a finite field of odd characteristic, and V a simple $\mathbb{F}G$-module induced from H. As mentioned, our next goal will be to achieve some control on the structure of $V{\downarrow_H}$.

Remark 9.5. Suppose that G is a solvable group, H is a non-normal subgroup of G having prime index, and L is the kernel of the permutation action of G (by right multiplication) on the set of right cosets modulo H. As permutation group on this set, G/L is then a Frobenius group of prime degree, with H/L as Frobenius complement (see [**8**, §16]); denote the Frobenius kernel of G/L by K/L. Note that K/L is a normal Sylow subgroup of prime order and H/L is a complementary Hall subgroup for it in G/L. Any two distinct conjugates of H/L have trivial intersection; moreover, each nontrivial subgroup of G/L either contains K/L or is contained in a unique conjugate of H/L. We shall also use that any transversal of L in K is a right transversal of H in G, that the permutation action of H on the set of nontrivial cosets of G modulo H matches the conjugation action on the nontrivial elements of K/L, and that the H-orbits of nontrivial cosets all have the same length, namely $|H/L|$ $(=|G/K|)$.

Lemma 9.6. *Let G be a solvable group, H a subgroup of G having odd prime index, $\mathbb{F}$ a finite field of odd characteristic, V a simple $\mathbb{F}G$-module, and W an absolutely simple submodule of $V{\downarrow_H}$ such that $V \simeq W{\uparrow^G}$. If W is not induced from the normal core L of H in G, then we have $V{\downarrow_H} \simeq (\bigoplus_{i=1}^{s} W) \oplus Y$, where s is an odd number and Y is a submodule of $V{\downarrow_H}$ such that $\operatorname{Hom}_{\mathbb{F}H}(W, Y) = 0$.*

Proof. First of all, H is definitely not a normal subgroup of G (otherwise W would be induced from $L = H$). Let X be a simple constituent of $W{\downarrow_L}$, and $I_G(X)$ its inertia subgroup in G (which certainly contains L). By 9.5, there are three cases to distinguish. First, if there exists an element g in $G \setminus H$ such that $L \leq I_G(X) \leq H^g$, then we get $I_H(X) = L$ and therefore W is induced by X from L, so this case can not arise. Second, we shall show that, if $L < I_G(X) \leq H$, then every simple submodule of $V{\downarrow_H}$ other than W itself has strictly larger dimension than W. In view of 8.1, this will prove our claim for this case (with $s = 1$). Start by noting that now $I := I_H(X) = I_G(X)$ and we have $I_H(X^g) = H \cap I^g = L$ for all g in $G \setminus H$. Consider a simple submodule T of $V{\downarrow_H}$, and suppose first that $T{\downarrow_L}$ has a simple constituent, say Z, isomorphic to X^h for some h in H. Then X^h and Z are in the same homogeneous component of $V{\downarrow_L}$ and this homogeneous component, call it U, is simple as $\mathbb{F}[I^h]$-module. The $\mathbb{F}[I^h]$-modules $W{\downarrow_{I^h}}$ and $T{\downarrow_{I^h}}$ both have nonzero intersection with the simple $\mathbb{F}[I^h]$-module U, so they both have to contain U. Thus the simple $\mathbb{F}H$-modules W and T have nonzero intersection, and this implies $W = T$. We are only interested in T if this does not hold, and now we know that then no simple constituent of $T{\downarrow_L}$ can be isomorphic to an X^h. Of course, any simple constituent Z of $T{\downarrow_L}$ (indeed, any simple constituent of $V{\downarrow_L}$) is isomorphic to X^g for some g in G; the conclusion from our argument so far is that $Z \simeq X^g$ for some g in $G \setminus H$. Thus $I_H(Z)$ is L, so that T is induced by Z from L. From this, we can see that the dimension of T is greater than the dimension of W: otherwise we would get $\dim W = \dim T = |H : L| \dim Z = |H : L| \dim X$, and therefore W would be induced by X from L.

We are left with the case in which $I_G(X)$ is not contained in any conjugate of H; the structure of G, as it was outlined in Remark 9.5, forces now $I_G(X)$ to contain the normal subgroup K (as defined in 9.5), so that K is of course contained in all the conjugates of $I_G(X)$ and therefore it stabilizes all the simple constituents of $W{\downarrow_L}$. In particular, since a transversal for H in G can be built up using only elements of K, we get $W^y{\downarrow_L}{\uparrow^H} \simeq (W{\downarrow_L})^y{\uparrow^H} \simeq W{\downarrow_L}{\uparrow^H}$ for all y in such a transversal. Now, considering the structure of G and taking in account the last comment, Mackey's Lemma yields

$$V{\downarrow_H} \simeq W \oplus \Big(\bigoplus_{i=1}^{n} W{\downarrow_L}{\uparrow^H} \Big)$$

where n is given by $(|G : H| - 1)/|H : L|$ (this follows again from the discussion in 9.5). Let α be a nonzero element in $\operatorname{Hom}_{\mathbb{F}H}(W, W{\downarrow_L}{\uparrow^H})$; $W\alpha$ is an $\mathbb{F}H$-submodule of $W{\downarrow_L}{\uparrow^H}$, in particular of $V{\downarrow_H}$, so that it is a direct summand in $W{\downarrow_L}{\uparrow^H}$. Denoting by S a direct complement for $W\alpha$ in $W{\downarrow_L}{\uparrow^H}$, again we consider $\mathrm{Hom}_{\mathbb{F}H}(W,S)$ and we iterate the process, eventually getting $W{\downarrow_L}{\uparrow^H} \simeq (\bigoplus_{i=1}^{t} W) \oplus R$, where R is an $\mathbb{F}H$-submodule of $W{\downarrow_L}{\uparrow^H}$ such that $\mathrm{Hom}_{\mathbb{F}H}(W,R) = 0$ (observe that t is now precisely the multiplicity of W in the socle of $W{\downarrow_L}{\uparrow^H}$). Therefore we get

$$V{\downarrow_H} \simeq \Big(\bigoplus_{i=1}^{nt+1} W \Big) \oplus Y$$

where Y is defined as $\bigoplus_{i=1}^{n} R$, and of course we have $\mathrm{Hom}_{\mathbb{F}H}(W,Y) = 0$. Our aim is now to show that nt is an even number. Certainly it is such if n is even. If n is odd then $|H:L|$ has to be even, and we shall see that in this case t turns out to be even.

Assume then n odd, and consider the representation of H/L which maps a generator to -1 in $\mathbb{F}$; we claim that, if X denotes an $\mathbb{F}H$-module associated to this representation, then $W \otimes X$ is isomorphic to W. Indeed, by Lemma 9.1, $W \otimes X$ and W have the same multiplicities as composition factors in the socle of $W{\downarrow_L}{\uparrow^H}$. If they are assumed to be nonisomorphic, this implies

$$V{\downarrow_H} \simeq \Big(\bigoplus_{i=1}^{nt+1} W \Big) \oplus Y \simeq \Big(\bigoplus_{i=1}^{nt} (W \otimes X) \Big) \oplus Y',$$

where $\mathrm{Hom}_{\mathbb{F}H}(W,Y) = \mathrm{Hom}_{\mathbb{F}H}(W \otimes X, Y') = 0$. But now, recalling that $\mathrm{End}_{\mathbb{F}H}(W)$ and $\mathrm{End}_{\mathbb{F}H}(W \otimes X)$ are isomorphic vector spaces, Lemma 8.1 gives

$$nt + 1 = |\,\mathrm{End}_{\mathbb{F}G}(V) : \mathrm{End}_{\mathbb{F}H}(W)| = |\,\mathrm{End}_{\mathbb{F}G}(V) : \mathrm{End}_{\mathbb{F}H}(W \otimes X)| = nt,$$

a clear contradiction. We are now in a position to apply Lemma 9.4 (as of course the kernel of X has index 2 in G), and the proof is complete. ∎

The next step shall be to remove the hypothesis of absolute irreducibility for W; this is done in the following theorem.

Theorem 9.7. *Let G be a solvable group, H a subgroup of G having odd prime index, $\mathbb{F}$ a finite field of odd characteristic, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow_H}$ such that $V \simeq W{\uparrow^G}$. If W is not induced from the normal core L of H in G, then we have $V{\downarrow_H} \simeq (\bigoplus_{i=1}^{s} W) \oplus Y$, where s is an odd number and Y is a submodule of $V{\downarrow_H}$ such that $\mathrm{Hom}_{\mathbb{F}H}(W,Y) = 0$.*

Proof. Let us denote by $\mathbb{E}$ the field $\mathrm{End}_{\mathbb{F}G}(V)$, and by $\mathbb{K}$ the field $\mathrm{End}_{\mathbb{F}H}(W)$. As mentioned in Section 6, a structure of $\mathbb{E}G$-module can be defined on V by considering the natural action of $\mathbb{E}$; similarly, W acquires a structure of $\mathbb{K}H$-module if we let $\mathbb{K}$ act naturally on it. Recall that the modules which arise in this way are denoted by $V_{\mathbb{E}}$ and $W_{\mathbb{K}}$ respectively. Also, $\mathbb{K}$ is embedded in $\mathbb{E}$, and therefore the underlying vector space of V can be endowed in a natural fashion with a structure of $\mathbb{K}G$-module. Denoting this module by $V_{\mathbb{K}}$, it is clear that $W_{\mathbb{K}}$ is a submodule of $(V_{\mathbb{K}}){\downarrow_H}$. Next, $V_{\mathbb{K}}$ is a simple $\mathbb{K}G$-module, and $W_{\mathbb{K}}$ an absolutely simple $\mathbb{K}H$-module; moreover, we have

$$\dim_{\mathbb{K}}(V_{\mathbb{K}}) = \frac{\dim_{\mathbb{F}} V}{|\mathbb{K}:\mathbb{F}|} = \frac{|G:H| \dim_{\mathbb{F}} W}{|\mathbb{K}:\mathbb{F}|} = |G:H| \dim_{\mathbb{K}}(W_{\mathbb{K}}),$$

whence we get $V_{\mathbb{K}} \simeq (W_{\mathbb{K}})\uparrow_H^G$. We claim now that $W_{\mathbb{K}}$ is not induced from L. Assume $W_{\mathbb{K}} \simeq X\uparrow_L^H$, where X is a submodule of $(W_{\mathbb{K}})\downarrow_L$. Certainly W contains $X_{\mathbb{F}}$ as an $\mathbb{F}H$-submodule and, since $\dim_{\mathbb{F}}(X_{\mathbb{F}}) = \dim_{\mathbb{K}}((X_{\mathbb{F}})^{\mathbb{K}}) = |\mathbb{K} : \mathbb{F}| \dim_{\mathbb{K}} X$ holds, we have

$$\dim_{\mathbb{F}} W = |\mathbb{K} : \mathbb{F}| \dim_{\mathbb{K}}(W_{\mathbb{K}}) = |\mathbb{K} : \mathbb{F}||H : L| \dim_{\mathbb{K}} X = |H : L| \dim_{\mathbb{F}}(X_{\mathbb{F}}),$$

so that W is induced by $X_{\mathbb{F}}$ from L, a contradiction.

We can finally apply Lemma 9.6, getting

$$(V_{\mathbb{K}})\downarrow_H \simeq \Big(\bigoplus_{i=1}^{s} W_{\mathbb{K}}\Big) \oplus Y,$$

where s is odd, and Y is a submodule of $(V_{\mathbb{K}})\downarrow_H$ such that $\mathrm{Hom}_{\mathbb{K}H}((W_{\mathbb{K}}), Y) = 0$. Now Lemma 8.1 yields $|\,\mathrm{End}_{\mathbb{K}G}(V_{\mathbb{K}}) : \mathrm{End}_{\mathbb{K}H}(W_{\mathbb{K}})| = s$ and, since $\mathrm{End}_{\mathbb{K}G}(V_{\mathbb{K}})$ is easily seen to be $\mathbb{E}$, another application of Lemma 8.1 leads to the desired conclusion. ∎

It is worth remarking that, in view of the discussion in [**5**, Section 7], Theorem 9.7 can be extended with no difficulties to modules over *not necessarily finite* fields of odd characteristic.

If, in the setting of Theorem 9.7, we drop the assumption of W not being induced from L, then we do loose control on the multiplicity of W in the socle of $V\downarrow_H$ (see Section 11). At any rate, if W is assumed to be induced from L, we also have a good understanding of the structure of $V\downarrow_H$. This situation is considered in the following lemma.

Lemma 9.8. *Let G be a solvable group, H a maximal subgroup of G having odd index, and $\mathbb{F}$ a field; let V be a simple $\mathbb{F}G$-module and W a submodule of $V\downarrow_H$ such that $V \simeq W\uparrow^G$. If W is induced from the normal core L of H in G, then $V\downarrow_H$ is semisimple, and all its simple submodules have dimension equal to $\dim W$. In particular, the composition length of $V\downarrow_H$ is an odd number.*

Proof. By the solvability of G and the maximality of H, it is possible to find a normal subgroup K of G such that $HK = G$ and $H \cap K = L$. Consider now a submodule Y of $W\downarrow_L$ such that W is induced by Y from L; we have

$$V \simeq (Y\uparrow^H)\uparrow^G \simeq (Y\uparrow^K)\uparrow^G .$$

If we view $V\downarrow_H$ as $[(Y\uparrow^K)\uparrow^G]\downarrow_H$, then Mackey's Lemma yields

$$V\downarrow_H \simeq ((Y\uparrow^K)\downarrow_L)\uparrow^H .$$

Chosen a right transversal T for L in K (of course the cardinality of T is $|G : H|$), we get now

$$V\downarrow_H \simeq \bigoplus_{g\in T} [(Y^g)\uparrow^H].$$

Since V is simple and the $(Y^g)\uparrow^H$ all have minimal dimension, V is induced from H by each of those; in particular they are all simple, so that $V\downarrow_H$ is semisimple with all the simple submodules having the same (minimal) dimension. ∎

Before going through the second fundamental result of this section, which is Theorem 9.10, we need a general lemma on symplectic modules.

Lemma 9.9. *Let H be a group, $\mathbb{F}$ a finite field, V a semisimple homogeneous $\mathbb{F}H$-module, and f an H-invariant nonsingular symplectic $\mathbb{F}$-form on V. If the composition length of V is odd, then there exists a simple submodule of V on which f does not vanish.*

Proof. We proceed by induction on the composition length of V, which we denote by $l_H(V)$ and write as $2k+1$ (k a nonnegative integer). If k is 0, then the claim is certainly true. Assume then $k > 0$, and choose a submodule U of V on which f vanishes and which is maximal subject to satisfy this property. We have $U \leq U^{\perp}$, and since (by a basic property of symplectic or symmetric forms) $l_H(V) = l_H(U) + l_H(U^{\perp})$ holds, one among $l_H(U)$ and $l_H(U^{\perp})$ is even and the other is odd. This ensures that U is properly contained in $U^{\perp}$ and, denoting by R a complement for U in $U^{\perp}$, we see that $l_H(R) = l_H(U^{\perp}) - l_H(U)$ is also an odd number, strictly smaller than $l_H(V)$. If f is singular on R then, setting $D := R \cap R^{\perp}$, we have that $U \oplus D$ is a submodule of V on which f vanishes. Since U is strictly contained in $U \oplus D$, and the latter is strictly contained in V, this contradicts the hypothesis of maximality on U. We conclude that f is nonsingular on R, and now the claim follows by induction. ■

Observe that the assumption of odd composition length for V is really needeed in the previous statement. In fact, given any $\mathbb{F}H$-module U, it is easy to see that $U \oplus U^*$ carries an H-invariant nonsingular symplectic $\mathbb{F}$-form. Therefore, as soon as U is a self-contragredient simple $\mathbb{F}H$-module of odd dimension, it produces a counterexample for the lemma above without the odd composition length assumption.

Theorem 9.10. *Let G be a solvable group, H a maximal subgroup of G having odd index, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module which carries a G-invariant nonsingular symplectic $\mathbb{F}$-form f, and W a submodule of $V{\downarrow_H}$ such that $V \simeq W{\uparrow^G}$. Assume that W is induced from the normal core L of H in G. Then there exists a submodule Z of $V{\downarrow_H}$ such that f does not vanish on Z, $V \simeq Z{\uparrow^G}$, and $|\operatorname{End}_{\mathbb{F}G}(V) : \operatorname{End}_{\mathbb{F}H}(Z)|$ is an odd number.*

Proof. Since, by Lemma 9.8, $V{\downarrow_H}$ is semisimple and its composition length is an odd number, there exists an odd number d such that an odd number of homogeneous components in $V{\downarrow_H}$ have composition length equal to d. Since V carries a G-invariant nonsingular symplectic $\mathbb{F}$-form, we have that V is self-contragredient, and of course the same holds for $V{\downarrow_H}$. Duality induces a permutation of order 2 on the set of homogeneous components of $V{\downarrow_H}$ and, since it preserves the dimensions, it permutes indeed the set of homogeneous components *with composition length d*. But now, since that set contains an odd number of elements, the relevant permutation has to fix some element in it; we conclude that there exists a homogeneous component X in $V{\downarrow_H}$ such that X is self-contragredient, and the composition length of X is odd. Of course any simple submodule of X is also self-contragredient and, as we shall see, this implies that the form f is nonsingular on X. Indeed, let R be the unique complement for X in $V{\downarrow_H}$; given an element y in X, the restriction to R of the map $y\psi_f$ is an element of R^* (recall that $y\psi_f$ maps an element v in V to $f(v,y)$), and the map $y \mapsto (y\psi_f){\downarrow_R}$ provides a morphism of $\mathbb{F}H$-modules (call it β) from X to R^*. If f is assumed singular on X, then there exists a nonzero element x in $X \cap X^{\perp}$ (orthogonality is meant with respect to f), and clearly we have $x\beta \neq 0$, otherwise x would lie in $V \cap V^{\perp}$ which is zero; we conclude that, decomposing X and R^* into the direct sum of simple submodules, say $X = U_1 \oplus \cdots \oplus U_d$ and $R^* = S_1 \oplus \cdots \oplus S_l$, there exist i in $\{1, \ldots, d\}$ and j in $\{1, \ldots, l\}$ such that the composite map $\beta{\downarrow_{U_i}}\, p_{S_j}$ is not the zero map (here p_{S_j} denotes the projection of R^* on S_j). But now U_i is isomorphic to S_j, and this is a contradiction because R^* does not contain any simple submodule isomorphic to U_j^*.

Now, by Lemma 9.9, there exists a simple submodule Z of X on which f does not vanish. Moreover, Z induces V from H (because of its dimension) and, by Lemma 8.1, we also have $|\mathrm{End}_{\mathbb{F}G}(V) : \mathrm{End}_{\mathbb{F}H}(Z)| = d$, an odd number. ■

It is worth mentioning that Theorem 9.10 remains true also without the assumption of maximality for H. This can be proved (once the result *with* the hypothesis of maximality has been established) arguing by induction on the index of H in G. As no direct use of such a generalization is made in this paper, the proof is omitted.

10. The main results

We are now in a position to prove Theorems 10.1 and 10.2, which are the main results of this paper. They provide a positive answer for the weak versions of Conjectures 4.3 and 4.1 (respectively) with some additional assumptions.

Theorem 10.1. *Let G be a solvable group, H a subgroup of G having odd prime index, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow_H}$. Assume also that V carries a G-invariant nonsingular symplectic $\mathbb{F}$-form f which does not vanish on W. If V is induced by W from H, then V is also form-induced from H (with respect to f).*

Proof. If the characteristic of $\mathbb{F}$ is 2, then we are done by Proposition 7.2(a) together with Remark 7.5; in this case, a submodule of $V{\downarrow_H}$ which form-induces V can be chosen isomorphic to W.

Assume now that the characteristic of $\mathbb{F}$ is odd. If W *is not* induced from the normal core of H in G, then we can apply Theorem 9.7, obtaining that the homogeneous component of W in the socle of $V{\downarrow_H}$ has odd composition length. Now, by Lemma 8.1, we have that $|\mathrm{End}_{\mathbb{F}G}(V) : \mathrm{End}_{\mathbb{F}H}(W)|$ is an odd number, and Lemma 7.7 (again together with Remark 7.5) yields the desired conclusion. Observe that even in this case a submodule of $V{\downarrow_H}$ which form-induces V can be chosen isomorphic to W. Finally, If W *is* induced from the normal core of H in G, then we can not guarantee that there exists a submodule of $V{\downarrow_H}$ isomorphic to W which form-induces V (see Example 11.1), but Theorem 9.10 (with 7.7 and 7.5, as usual) leads anyway to the conclusion. ∎

Finally, we go back to our original problem.

Theorem 10.2. *Let G be a solvable group, H a subgroup of G having odd prime index, and D a faithful, primitive, tensor-indecomposable representation of G. Assume that we have $\bar{D}{\downarrow_H} \simeq \bar{P}_1 \otimes \bar{P}_2$, where P_1 and P_2 are projective representations of H. If $\deg P_2$ is not 1, and $(\deg P_2)^{|G:H|}$ is a divisor of $\deg D$, then we have $(\deg P_2)^{|G:H|} = \deg D$, and there exists a projective representation P of H such that $\bar{D} \simeq \bar{P}{\uparrow^{\otimes G}}$ holds.*

Proof. As in the proof of 8.3, we apply first Lemma 4.2. Then the claim follows from Theorem 10.1 together with Theorem 3.7. ∎

We conclude the section showing that the weak version of Conjecture 4.1 provides (in the cases in which it is confirmed) a good test for deciding whether a representation is tensor induced or not from a given subgroup.

Theorem 10.3. *Let G be a group with noncentral Fitting subgroup F, and D a faithful, quasi-primitive, tensor-indecomposable representation of G. Assume that H is a subgroup of G for which the weak version of Conjecture 4.1 holds. Then the following conditions are equivalent.*

(a) *There exists a projective representation P of H such that $\bar{D} \simeq \bar{P}{\uparrow^{\otimes G}}$.*

(b) *H contains F, and there exists a subgroup K of F such that K is normal in H, $Z(K) = Z(F)$, and $|F/K|^{|G:H|} = (\deg D)^2$.*

Proof. If condition (a) holds, then H contains F by 3.5(b); moreover, there exists a projective representation Q of H such that $\bar{D}{\downarrow_H} \simeq \bar{P} \otimes \bar{Q}$, and the tensor-indecomposability of D (together with the quasi-primitivity) ensures that $D{\downarrow_F}$ is irreducible. We can therefore apply Theorem 2.2 and Lemma 2.1(b) (observe that F satisfies the hypotheses of 2.1), obtaining that the kernel of $\bar{P}{\downarrow_F}$ has the required properties. Conversely, if (b) holds, applying 2.2 we see that there exist projective representations X and Y of H such that $\bar{D}{\downarrow_H} \simeq \bar{X} \otimes \bar{Y}$, and $\ker(\bar{X}{\downarrow_F}) = K$; also, the hypothesis of Conjecture 4.1 concerning the degrees is satisfied by D and X (this follows from Lemma 2.1(b)), and an application of the weak version of 4.1 leads to the desired conclusion. ∎

11. An example

The example presented in this section shows that the situation outlined in Lemma 7.7 can actually occur *even if the index of the subgroup H is an odd prime.* In other words, the relationship between induction and form induction can be rather awkward (in the sense that induction by an anisotropic H-submodule W *does not imply* form induction by an H-submodule isomorphic to W), and not only when an 'even step' is involved (as in Example 5.2).

Another question which may arise naturally is whether we really need to assume, in the statement of Theorem 9.7, that W is not induced from the normal core of H in G. As we shall see in a moment, the answer is 'yes'.

Example 11.1. Consider a semidirect product $G := A_4 \rtimes Q_{16}$ of the alternating group on 4 objects and the quaternion group of order 16. The two groups in question are generated as follows:

$$A_4 = (C_2 \times C_2) \rtimes C_3 = \langle a, a^{b^2}, b \rangle$$

(here a, a^{b^2} are generators of of $C_2 \times C_2$, and b is a generator of C_3), and

$$Q_{16} = \langle c, d \ : \ c^4 = d^2 = m \ , \ m^2 = 1 \ , \ c^d = c^{-1} \rangle;$$

also, the action of Q_{16} on A_4 is defined by $a^c = aa^{b^2}$, $(a^{b^2})^c = a^{b^2}$, $b^c = b^2$ (the action of d is trivial). Let us consider the subgroups

$$H := \langle a, a^{b^2}, c, d \rangle \simeq (C_2 \times C_2) \rtimes Q_{16},$$

and $L := \langle a, a^{b^2}, c^2, d \rangle \simeq C_2 \times C_2 \times Q_8$, which is the normal core of H in G. We see that $L/\langle a, a^{b^2}c^4 \rangle$ is isomorphic to Q_8; therefore, denoting by $\mathbb{F}$ the prime field in characteristic 3, it is possible to define a 2-dimensional simple $\mathbb{F}L$-module Y on which the elements c^2 and d (whose cosets modulo $\langle a, a^{b^2}c^4 \rangle$ generate $L/\langle a, a^{b^2}c^4 \rangle$) act respectively (on the right) as the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Denoting by W the $\mathbb{F}H$-module $Y{\uparrow}^H$, it is routine to check that W is absolutely simple. Next, consider the simple $\mathbb{F}G$-module $V := W{\uparrow}^G$. The restriction $V{\downarrow}_H$ is certainly semisimple, and it can be shown that $V{\downarrow}_H \simeq W \oplus W \oplus Z$, where the simple constituent Z is not isomorphic to W (whence *the assumption that W is not induced from the normal core of H in G is really needed in Theorem* 9.7*, and even in Lemma* 9.6). We also observe that Z, which induces V from H as well as W, is not absolutely simple (its endomorphism ring is the finite field with 3^2 elements). Moreover, $Z{\downarrow}_L$ is homogeneous, whereas $W{\downarrow}_L$ is not.

Now, the $\mathbb{F}L$-module Y is endowed with an L-invariant nonsingular symplectic $\mathbb{F}$-form (the action of L on Y is given by two matrices which lie in $Sp(2,3)$), so that W carries an H-invariant nonsingular symplectic $\mathbb{F}$-form. Since the multiplicity of W in $V{\downarrow}_H$ is 2, Lemma 8.1 yields that $\mathrm{End}_{\mathbb{F}G}(V)$ has even degree (namely 2) as a field extension of $\mathrm{End}_{\mathbb{F}H}(W)$. Finally, any simple module for G (over an arbitrary field) turns out to be self-contragredient because every element of G is conjugate to its inverse, so Theorem 6.6 ensures that τ_V is the identity on $\mathrm{End}_{\mathbb{F}G}(V)$. In other words, *we are in the situation of Lemma* 7.7(a)*, in a case in which the subgroup H has index* 3.

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